Satisfiability Checking

Fourier-Motzkin Variable Elimination

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The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

$$
\begin{aligned}
(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0) \wedge p_1 + p_2 + p_3 \geq 100 \wedge \\
(p_1 \geq 5 \vee p_2 \geq 5) \wedge p_3 \geq 10 \wedge p_1 + 2p_2 + 5p_3 \leq 180 \wedge \\
3p_1 + 2p_2 + p_3 \leq 300
\end{aligned}
$$

Linear arithmetic over the reals

- Goal: decide satisfiability of
  conjunctions of linear constraints over the reals

$$\bigwedge_{1 \leq i \leq m} \sum_{1 \leq j \leq n} a_{ij} x_j \leq b_i$$

- Input in matrix form: $A\overline{x} \leq \overline{b}$

m constraints
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{22} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{pmatrix}$$
n variables

Fourier-Motzkin variable elimination

- Earliest method for solving linear inequalities
  discovered in 1826 by Fourier, re-discovered by Motzkin in 1936

- Basic idea of variable elimination:
  - Pick a variable and eliminate it, yielding an equisatisfiable formula that does not refer to the eliminated variable any more.
  - Continue until all variables are eliminated.

- Fourier-Motzkin: Put requirements on the lower an upper bounds on the variable we want to eliminate.

Variable bounds

- For a variable x_n, we can partition the constraints according to the coefficient a_{in}:
  - $a_{in} > 0$: upper bound β_i on x_n
  - $a_{in} < 0$: lower bound β_i on x_n

$$\sum_{j=1}^{n} a_{ij} \cdot x_j \leq b_i$$

$$\Rightarrow \quad a_{in} \cdot x_n \ \leq \ b_i - \sum_{j=1}^{n-1} a_{ij} \cdot x_j$$

$$(a) \quad \overset{a_{in}>0}{\Rightarrow} \quad x_n \ \leq \ \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \quad =: \beta_l \quad \text{upper bound}$$

$$(b) \quad \overset{a_{in}<0}{\Rightarrow} \quad x_n \ \geq \ \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \quad =: \beta_u \quad \text{lower bound}$$

Example for upper and lower bounds

		Category for x_1?
(1)	$x_1 - x_2 \leq 0$	Upper bound
(2)	$x_1 - x_3 \leq 0$	Upper bound
(3)	$-x_1 + x_2 + 2x_3 \leq 0$	Lower bound
(4)	$-x_3 \leq -1$	No bound

Eliminating unbounded variables

- Iteratively remove variables that are not bounded in both ways (and all the constraints that use them).
- The new problem has a solution iff the old problem has one!

$$\begin{array}{rcl} \sout{8x} & \sout{\geq} & \sout{7y} \\ \sout{x} & \sout{\geq} & \sout{3} \\ y & \geq & z \\ z & \geq & 10 \\ 20 & \geq & z \end{array} \longrightarrow \begin{array}{rcl} \sout{y} & \sout{\geq} & \sout{z} \\ z & \geq & 10 \\ 20 & \geq & z \end{array} \longrightarrow \begin{array}{rcl} z & \geq & 10 \\ 20 & \geq & z \end{array}$$

Fourier-Motzkin variable elimination

- For each pair of a lower bound β_l and an upper bound β_u, we have

$$\beta_l \quad \leq \quad x_n \quad \leq \quad \beta_u$$

- For each such pair, add the constraint

$$\beta_l \quad \leq \quad \beta_u$$

			Category for x_1?
~~(1)~~	~~$x_1 - x_2 \leq 0$~~		Upper bound
~~(2)~~	~~$x_1 - x_3 \leq 0$~~		Upper bound
~~(3)~~	~~$-x_1 + x_2 + 2x_3 \leq 0$~~		Lower bound
(4)	$-x_3 \leq -1$		Lower bound
			eliminate x_1
(5)	$2x_3 \leq 0$	(from 1,3)	Upper bound
(6)	$x_2 + x_3 \leq 0$	(from 2,3)	Upper bound
			we eliminate x_3
(7)	$1 \leq 0$	(from 4,5)	

$\rightarrow$ Contradiction (the system is UNSAT)

Complexity

- Worst-case complexity:

$$m \to m^2 \to (m^2)^2 \to \ldots \to m^{2^n}$$

- Heavy!

- The bottleneck: case-splitting