Satisfiability Checking

Decidability and Decision Procedures

–Some Historical Notes–

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FO theories and their decidability

Some first-order theories:

Logic	decidability	algorithm
Propositional logic	decidable	SAT-solving
Equational logic	decidable	SAT-encoding
Equational logic with uninterpr. functions	decidable	SAT-encoding
Linear real algebra ($\mathbb{R}$ with $+$)	decidable	Simplex
Real algebra ($\mathbb{R}$ with $+$ and $*$)	decidable	CAD
		virtual substitution
		Gröbner bases
Presburger arithmetic ($\mathbb{N}$ with $+$)	decidable	branch and bound,
		Omega test
Peano arithmetic ($\mathbb{N}$ with $+$ and $*$)	undecidable	-

But actually what does it mean "decidable" or "undecidable"?

Milestones for Peano arithmetic

- Peano arithmetic goes all the way back to ancient Greek mathematics.
- But the modern theory of arithmetic was developed only in the second half of the 19th century.
- Hermann Graßmann (1809-1877)
- Richard Dedekind (1833-1916)
- Gottlob Frege (1848-1925)
- Giuseppe Peano (1858-1932)
- ...

Hermann Graßmann (1809-1877)

- "Die lineare Ausdehnungslehre, ein neuer Zweig der Mathematik" [The Theory of Linear Extension, a New Branch of Mathematics] (1844)
- Basics of linear algebra and vector spaces.
- Grassmann showed that once geometry is put into the algebraic form, then the number 3 has no privileged role as the number of spatial dimensions; the number of possible dimensions is in fact unbounded.

Richard Dedekind (1833-1916)

"Stetigkeit und irrationale Zahlen" [Continuity and irrational numbers] (1912)

- Dedekind cut: An irrational number divides the rational numbers into two sets, with all the members of one set (upper) being strictly greater than all the members of the other (lower) set.
- Every location on the number line continuum contains either a rational or an irrational number. Thus there are no empty locations, gaps, or discontinuities.

Richard Dedekind (1833-1916)

- If there existed a one-to-one correspondence between two sets, Dedekind said them to be "similar". He invoked similarity to give the first precise definition of an infinite set:

Definition (Dedekind's theorem)

A set is infinite when it is "similar to a proper part of itself".

E.g., the set $\mathbb{N}$ of natural numbers can be shown to be similar to the subset of square numbers.

- He also proposed an axiomatic foundation for the natural numbers, one year before Peano formulated an equivalent but simpler set of axioms, now the standard ones.

Gottlob Frege (1848-1925)

- “Begriffsschrift” (1879)
- Rigorous treatment of functions and variables.
- Frege invented axiomatic predicate logic, especially quantified variables. Previous logic had dealt with the logical operators “and”, “or”, “some”, “all” etc., but iterations of these operations, especially “some” and “all”, were little understood.

Gottlob Frege (1848-1925)

- One of Frege's stated purposes was to isolate the logical principles of inference, so that in the proper representation of mathematical proof, one would at no point appeal to "intuition". If there was an intuitive element, it was to be isolated and represented separately as an axiom; from there on, the proof was to be purely logical and without gaps.
- Having exhibited this possibility, Frege's larger purpose was to defend the view that arithmetic is a branch of logic: unlike geometry, arithmetic was to be shown to have no basis in "intuition", and no need for non-logical axioms.
- The analysis of logical concepts and the machinery of formalization that is essential to the incompleteness results of Gödel and Turing is ultimately due to Frege.

Giuseppe Peano (1858-1932)

“The principles of arithmetic, presented by a new method” (1889)

- Influence of Dedekind
- Axiomatic theory of arithmetic

Peano axioms

Axiomatize Peano arithmetic, i.e.,

- the set $\mathbb{N}$ of natural numbers,
- the operator $+$ (successor function S), and
- the operator $*$.

Peano axioms (first-order form)

- A1 $0 \in \mathbb{N}$
- A2 $\forall n \in \mathbb{N}.\ n + 1 \in \mathbb{N}$
- A3 $\forall n \in \mathbb{N}.\ n + 1 \neq 0$
- A4 $\forall n, m \in \mathbb{N}.\ n + 1 = m + 1 \rightarrow n = m$
- A5 $\forall n \in \mathbb{N}.\ n + 0 = n$
- A6 $\forall n, m \in \mathbb{N}.\ n + (m + 1) = (n + m) + 1$
- A7 $\forall n \in \mathbb{N}.\ n * 0 = 0$
- A8 $\forall n, m \in \mathbb{N}.\ n * (m + 1) = n * m + n$
- A9 For every $k \in \mathbb{N}$ and every Peano formula $\varphi(x_0, \ldots, x_k)$ an instance of the following first-order induction schema:

$$\forall \vec{m} \in \mathbb{N}^k.\ [(\varphi(0, \vec{m}) \wedge$$
$$[\forall n \in \mathbb{N}.\ (\varphi(n, \vec{m}) \rightarrow \varphi(n + 1, \vec{m}))]) \rightarrow (\forall n.\ \varphi(n, \vec{m}))]$$

Note: there are infinitely many axioms in the second-order form!

Q: Prove $5 \in \mathbb{N}$! Prove $\forall n.\ n \neq n + 1$!

- When the Peano axioms were first proposed, Bertrand Russell and others agreed that these axioms implicitly defined what we mean by a "natural number".
- Henri Poincaré was more cautious, saying they only defined natural numbers if they were consistent; if there is a proof that starts from just these axioms and derives a contradiction such as $0 = 1$, then the axioms are inconsistent, and don't define anything.
- International Congress of Mathematicians at Paris in 1900: David Hilbert posed the problem of proving their consistency using only finitistic methods as the second of his 23 problems.

David Hilbert

- Researchers' primary aim should be to establish mathematics on a solid and provably consistent foundation of axioms, from which, in principle, all mathematical truths could be deduced (by the standard methods of predicate logic).
- Entscheidungsproblem (decision problem): Could an effective procedure be devised which would demonstrate –in a finite time– whether any given mathematical proposition was, or was not, provable from a given set of axioms?

Consistency, completeness, and decidability

Here we can see three distinguishable concerns.

- Consistency: The set of axioms should be consistent, and provably so.
- Completeness: All mathematical truths should be deducible from those axioms.
- Decidability: There should be a clearly formulated procedure which is such that, given any statement of mathematics, it can definitively establish within a finite time whether or not that statement follows from the given axioms.

Consistency vs completeness

- A consistent system is one in which it is never possible to prove both a proposition P and its negation $\neg P$.
- A complete system is one in which it is always possible to prove either P or $\neg P$, for any proposition P that is expressible within the system.

If we were able to achieve a consistent and complete system of arithmetic, with true axioms and valid rules, then any arithmetical proposition would be provable if, and only if, it is true. A major part of Hilbert's dream would thus be realised.

Completeness of Presburger arithmetic

- Around 1920: Presburger proved that Presburger arithmetic is complete (using quantifier elimination).
- Would multiplication make the difference?
- 1929: Thoralf Skolem showed that the theory of $\mathbb{N}$ with $*$ but without the successor function and $+$ is complete.
- Around 1930: Alfred Tarski showed completeness of real algebra.
- So Peano arithmetic was expected to be complete, too.
- (In 1950 Raphael M. Robinson will show that Peano arithmetic without induction, called Robinson arithmetic, is complete.)

Kurt Gödel (1906-1978), first incompleteness theorem

In a famous paper published in 1931, Gödel proved his

Theorem (First incompleteness theorem)

- *In any true (and hence consistent) axiomatic theory*
- *sufficiently rich to enable the expression and proof of basic arithmetic propositions,*
- *it will be possible to construct an arithmetical proposition G such that neither G, nor its negation, is provable from the given axioms.*

Hence the system must be incomplete.
Moreover G must be a true statement of arithmetic.

Proof of Gödel's first incompleteness theorem

Proof.

- Gödel's proof ingeniously shows how statements about mathematical relationships (e.g. that a particular sequence of propositions provides a proof of some proposition P) can be encoded as statements within arithmetic.
- This encoding, moreover, is truth-preserving, so that the encoded "meta-mathematical" statement will be true if, and only if, the encoding statement of arithmetic is true.

□

Proof of Gödel's first incompleteness theorem (cont.)

Proof.

- Gödel derived an arithmetical proposition G which encodes the statement that G itself is unprovable within the system.
- Assume that the system would be complete.

  Completeness $\rightarrow$ G is false

  G is false $\rightarrow$ G is provable

  G is false $\rightarrow$ $\neg G$ is true $\stackrel{completeness}{\rightarrow}$ $\neg G$ is provable.
- Thus if the system is complete, it cannot be consistent, since both G and $\neg G$ would then be provable within it.

□

Gödel's second incompleteness theorem

Gödel's second incompleteness theorem, also published in the same article, follows from the first.

Theorem

No consistent axiomatic theory sufficiently rich to enable the expression and proof of basic arithmetic propositions can prove its own consistency.

Proof.

Suppose that a system was able to prove its own consistency. Then by the above argumentation G is provable within the system. But since G encodes the statement that G is unprovable within the system, we have a contradiction. It follows that the system cannot after all prove its own consistency. □

What remains open

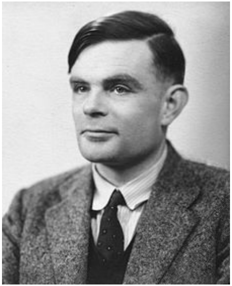
- Gödel's incompleteness theorems left the Entscheidungsproblem as unfinished business.
- He had shown that any consistent axiomatic system of arithmetic would leave some arithmetical truths unprovable (without any computable function).

Definition

A function is computable, if there is an algorithm that can calculate its result, in a finite number of steps.

- However, this did not in itself rule out the existence of some "effectively computable" decision procedure which would infallibly, and in a finite time, reveal whether or not any given proposition was, or was not, provable.

Alan Turing (1912-1954)



"On Computable Numbers, with an Application to the Entscheidungsproblem" (1936)

- He devised a rigorous notion of effective computability based on the "Turing Machine".

Effective methods, effective computability

Definition

An effective method is one which reduces the solution of some class of problems to a series of routine steps which

- always gives some answer after a finite time (termination),
- always gives the right answer (soundness), and
- works for all problem instances of the class (completeness).

- An effective method for calculating the values of a function is an algorithm.
- Functions with an effective method are sometimes called effectively computable.

Turing's theorem

"On Computable Numbers, with an Application to the Entscheidungsproblem" (1936):

- He devised a rigorous notion of effective computability based on the "Turing Machine".
- He then showed that there exist problems that cannot be effectively computed by this means.
- He did so by proving the impossibility of devising a Turing Machine program that can determine infallibly (and within a finite time) whether or not a given Turing Machine will eventually halt given some arbitrary input (Halting Problem).

Hence Turing proved that Hilbert's Entscheidungsproblem was unsolvable.

Alonzo Church (1903-1995)

Theorem (Church's Thesis)

If some algorithm exists to carry out a calculation, then the same calculation can also be carried out by a Turing machine (as well as by a recursively definable function).

- Church's thesis is not a mathematical statement and cannot be proven by a mathematical proof.
- Despite this fact, the Church–Turing thesis now has near-universal acceptance.

Hilbert's dream was shattered...

- Any consistent axiomatic theory sufficiently rich to enable the expression and proof of basic arithmetic propositions can be neither complete (as Gödel had shown) nor effectively decidable (by Turing).
- Paris and Harrington (1977) gave the first "natural" example of a statement which is true for the integers but unprovable in Peano arithmetic.