Exercise 1

Charlie Brown walks his dog Snoopy every day the same way:

• Both leave the house next to each other and start their walk.

• As Charlie ($C$) is thinking about important things (the girl with the red hair), he walks with continuous pace $v_c$.

• Curious Snoopy ($S$) is less steady and thus changes his pace between $-v_s$ and $v_s$, while $0 < v_c < v_s$ holds.

• The leash has only a length of 2 meters. Whenever Snoopy is left behind 2 meters, Charlie waits until Snoopy closes up to him and both continue the walk.

Please give a linear hybrid automaton, which models the position $S$ and $C$ of Snoopy and Charlie respectively.

Solution: The linear hybrid automaton which models the position of Charlie and Snoopy can be specified as follows:

\[
\begin{align*}
\text{walk} & : \begin{cases} 
\dot{C} = v_c \\
\dot{S} \in [-v_s, v_s] \\
-2 \leq C - S \leq 2
\end{cases} \\
\text{wait} & : \begin{cases} 
\dot{C} = 0 \\
\dot{S} \in [-v_s, v_s] \\
-2 \leq C - S \leq 2
\end{cases}
\end{align*}
\]

Exercise 2

We consider a vehicle platoon, where two cars are driving with speeds $\dot{x}_i \in [l, u], i \in \{1, 2\}, 0 < l < u$ on a road, such that the 1\textsuperscript{st} car is in front of the 2\textsuperscript{nd} car. The
goal is to keep the distance between two cars above some constant $d_0 > 0$. When the distance is at its boundary $d_0$, the rear car brakes, which limits its speed to the interval $\dot{x}_2 \in [l_{\text{min}}, l], 0 < l_{\text{min}} < l$. Additionally we utilize a second constant $d_1 > d_0 > 0$ to prolong the braking process until this target distance $d_1$ is reached. Initially the goal condition is satisfied.

Note that both transitions are *urgent transitions*, which means that they are taken as soon as they are enabled.

A linear hybrid automaton of the above system is given as follows:

\[ x_1 - x_2 \geq d_0 \]

\[ \dot{x}_1 \in [l, u] \]
\[ \dot{x}_2 \in [l, u] \]
\[ x_1 - x_2 \geq d_0 \]
\[ x_1 - x_2 = d_0 \]
\[ \dot{x}_1 \in [l, u] \]
\[ \dot{x}_2 \in [l_{\text{min}}, l] \]
\[ x_1 - x_2 \leq d_1 \]
\[ x_1 - x_2 = d_1 \]

a) Please calculate the forward time closure as presented in the lecture.

Solution:

a) We use the algorithm presented in the lecture, where $e_1 = (l_0, x_1^{\text{pre}} - x_2^{\text{pre}} = d_0, x =$
\(x^{\text{pre}} \land y = y^{\text{pre}}, l_1\) and \(e_2 = (l_1, x_1^{\text{pre}} - x_2^{\text{pre}} = d_1, x = x^{\text{pre}} \land y = y^{\text{pre}}, l_0)\):

\[P_0 = \{(l_0, R_0^0), (l_1, R_1^0)\}\]
\[R_0^0 = T_{l_0}^+(\varphi_{\text{init}} \land \varphi_{\text{inv}})\]
\[= T_{l_0}^+(x_1 - x_2 \geq d_0)\]
\[= \exists t. \exists x_1^{\text{pre}}. \exists x_2^{\text{pre}}. x_1^{\text{pre}} - x_2^{\text{pre}} \geq d_0 \land t \geq 0 \land \]
\[x_1 \leq x_1^{\text{pre}} + u \cdot t \land x_1 \geq x_1^{\text{pre}} + l \cdot t \land \]
\[x_2 \leq x_2^{\text{pre}} + u \cdot t \land x_1 \geq x_1^{\text{pre}} + l \cdot t \land \]
\[x_1 - x_2 \geq d_0\]

\((F.M.)\)
\[= \exists t. x_1^{\text{pre}}. t \geq 0 \land \]
\[x_1 \leq x_1^{\text{pre}} + u \cdot t \land x_1 \geq x_1^{\text{pre}} + l \cdot t \land \]
\[x_2 - u \cdot t \leq x_2 - l \cdot t \land x_1 - u \cdot t \leq x_1^{\text{pre}} - d_0 \land \]
\[x_1 - x_2 \geq d_0\]

\((F.M.)\)
\[= \exists t. t \geq 0 \land \]
\[x_2 - u \cdot t + d_0 \leq x_1 - l \cdot t \land \]
\[x_1 - u \cdot t \leq x_1 - l \cdot t \land \]
\[x_2 - u \cdot t \leq x_2 - l \cdot t \land \]
\[x_1 - x_2 \geq d_0\]

\((F.M.)\)
\[= x_1 - x_2 \geq d_0\]
\[R_1^0 = T_{l_1}^+(\varphi_{\text{init}} \land \varphi_{\text{inv}}) = \text{false}\]
\[P_1 = \{(l_0, R_0^1), (l_1, R_1^1)\}\]
\[R_1^1 = T_{l_1}^+(D^+(R_0^0))\]
\[R_1^1 = T_{l_1}^+(\exists x_1^{\text{pre}} \exists x_2^{\text{pre}}.\]
\[x_1^{\text{pre}} - x_2^{\text{pre}} \geq d_0 \land x_1^{\text{pre}} - x_2^{\text{pre}} = d_0 \land x_1 = x_1^{\text{pre}} \land x_2 = x_2^{\text{pre}} \land x_1^{\text{pre}} - x_2^{\text{pre}} \leq d_1)\]
\[= T_{l_1}^+(x_1 - x_2 = d_0 \land x_1 - x_2 \leq d_1)\]
\[ R_{l_1}^1 = \exists t. \exists x_{1\text{pre}}. \exists x_{2\text{pre}}. t \geq 0 \land \]
\[ x_{2\text{pre}} = x_{1\text{pre}} - d_0 \land \]
\[ x_1 \leq x_{1\text{pre}} + u \cdot t \land x_1 \geq x_{1\text{pre}} + l \cdot t \land \]
\[ x_2 \leq x_{2\text{pre}} + l \cdot t \land x_2 \geq x_{2\text{pre}} + l_{\text{min}} \cdot t \land \]
\[ x_1 - x_2 \leq d_1 \]
\[ = \exists t. \exists x_{1\text{pre}}. t \geq 0 \land \]
\[ x_1 \leq x_{1\text{pre}} + u \cdot t \land x_1 \geq x_{1\text{pre}} + l \cdot t \land \]
\[ x_2 \leq x_{1\text{pre}} - d_0 + l \cdot t \land x_2 \geq x_{1\text{pre}} - d_0 + l_{\text{min}} \cdot t \land \]
\[ x_1 - x_2 \leq d_1 \]
\[ = \exists t. \exists x_{1\text{pre}}. t \geq 0 \land \]
\[ x_1 - u \cdot t \leq x_{1\text{pre}} \land x_{1\text{pre}} \leq x_1 - l \cdot t \land \]
\[ x_2 + d_0 - l \cdot t \leq x_{1\text{pre}} \land x_{1\text{pre}} \leq x_2 + d_0 - l_{\text{min}} \cdot t \land \]
\[ x_1 - x_2 \leq d_1 \]
\[ \Rightarrow (F.M.) \exists t. t \geq 0 \land \]
\[ x_1 - u \cdot t \leq x_1 - l \cdot t \land x_1 - u \cdot t \leq x_2 + d_0 - l_{\text{min}} \cdot t \land \]
\[ x_2 + d_0 - l \cdot t \leq x_1 - l \cdot t \land x_2 + d_0 - l \cdot t \leq x_2 + d_0 - l_{\text{min}} \cdot t \land \]
\[ x_1 - x_2 \leq d_1 \]
\[ \Rightarrow (F.M.) \exists t. t \geq 0 \land \]
\[ \frac{x_1 - x_2 - d_0}{u - l_{\text{min}}} \leq t \land x_1 - x_2 \geq d_0 \land x_1 - x_2 \leq d_1 \]
\[ \Rightarrow x_1 - x_2 \geq d_0 \land x_1 - x_2 \leq d_1 \]
\[ R_{l_0}^2 = T_{l_0}^+(D_{e_0}^+(R_{l_1}^1)) \]
\[ = T_{l_0}^+(\exists x_{1\text{pre}}. \exists x_{2\text{pre}}. \land \]
\[ d_0 \leq x_1 - x_2 \land x_1 - x_2 \leq d_1 \land \]
\[ x_{1\text{pre}} - x_{2\text{pre}} = d_1 \land x_1 = x_{1\text{pre}} \land x_2 = x_{2\text{pre}} \land x_1 - x_2 \geq d_0) \]
\[ = T_{l_1}^+(d_0 \leq x_1 - x_2 \land x_1 - x_2 \leq d_1 \land x_1 - x_2 = d_1) \]
\[ = T_{l_1}^+(x_1 - x_2 = d_1) \subseteq R_{l_0}^0 \]

Reachable set: \( R_{l_0}^0 \cup R_{l_1}^1 \)