Modeling and Analysis of Hybrid Systems

Reachability analysis using Taylor models

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Informatik 2 - Theory of Hybrid Systems
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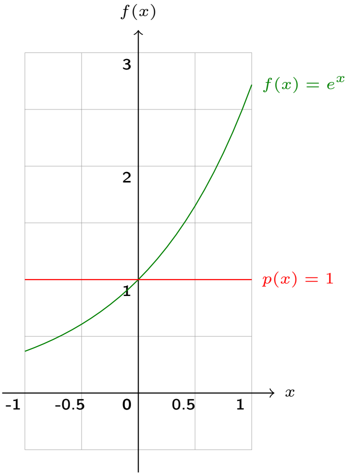
Higher-order approximations

Polynomial p is a k-order approximation of a smooth $f : D \to \mathbb{R} \ \in C^k$ iff $f(\mathbf{c}) = p(\mathbf{c})$ for the center point $\mathbf{c}$ of D and for each $0 < m \leq k$:

$$\left.\frac{\partial^m f}{\partial \mathbf{x}^m}\right|_{\mathbf{x}=\mathbf{c}} = \left.\frac{\partial^m p}{\partial \mathbf{x}^m}\right|_{\mathbf{x}=\mathbf{c}} .$$

Examples

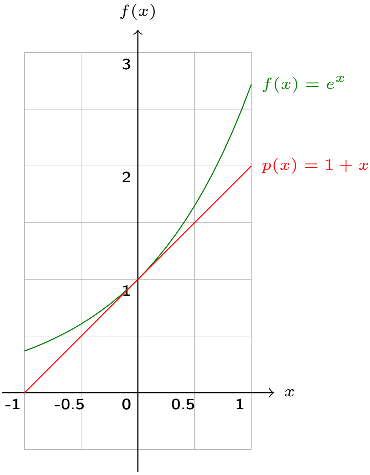
Several higher-order approximations of $f(x) = e^x$ with $x \in [-1, 1]$



0-order approximation

Examples

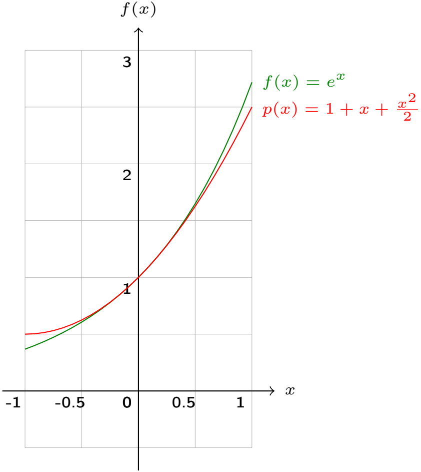
Several higher-order approximations of $f(x) = e^x$ with $x \in [-1, 1]$



1-order approximation

Examples

Several higher-order approximations of $f(x) = e^x$ with $x \in [-1, 1]$



2-order approximation

Taylor models

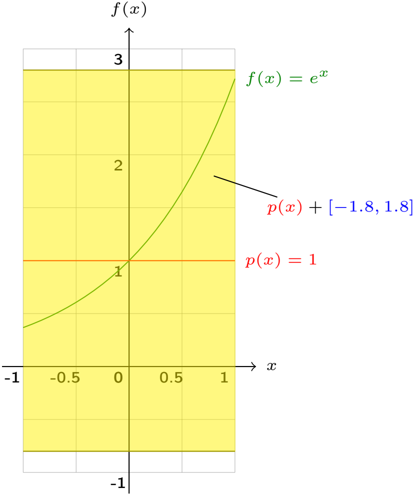
- Taylor model: a pair (p, I) over an interval domain D
- It defines the set

$$\{\mathbf{x} = p(\mathbf{x}_0) + \mathbf{y} \mid \mathbf{x}_0 \in D \wedge \mathbf{y} \in I\}$$

- Taylor model arithmetic: Closed under many basic operations

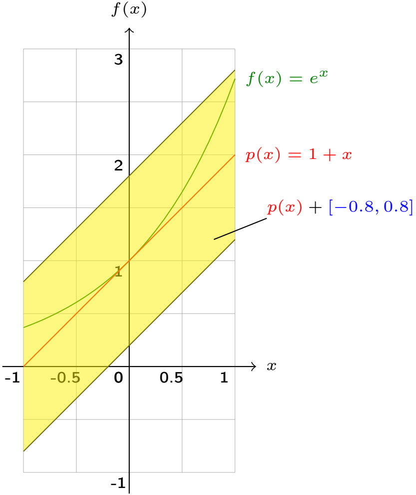
[Berz et al., 1999]

Higher-order over-approximations by Taylor models



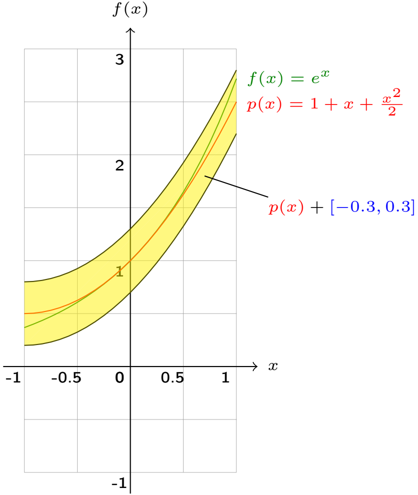
0-order over-approximation

Higher-order over-approximations by Taylor models



1-order over-approximation

Higher-order over-approximations by Taylor models



2-order over-approximation

Verified integration by Taylor models

Given:

- ODE: $\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, t)$
- Initial set: Taylor model X_0 over domain D_0

[Berz et al., 1999]

Verified integration by Taylor models

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Integration step for time $[0, \delta]$:

[Berz et al., 1999]

Verified integration by Taylor models

Given:

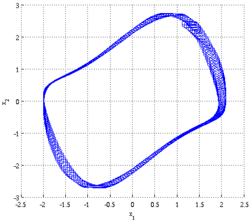
- ODE: $\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, t)$
- Initial set: Taylor model X_0 over domain D_0

Integration step for time $[0, \delta]$:

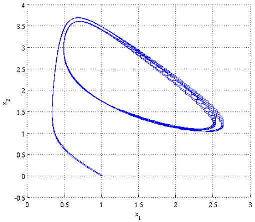
1. Compute a k-order approximation $p_k(\mathbf{x}_0, t)$ over $X_0 \times [0, \delta]$
2. Determine remainder I_k such that $(p_k(\mathbf{x}_0, t), I_k)$ over $X_0 \times [0, \delta]$ over-approximates the flow pipe segment
3. Initial set for the next flowpipe segment: $(p_k(\mathbf{x}_0, \delta), I_k)$ over X_0

[Berz et al., 1999]

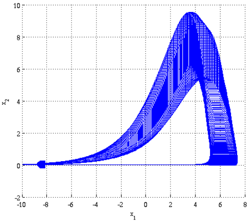
Continuous systems:



Van-der-Pol oscillator

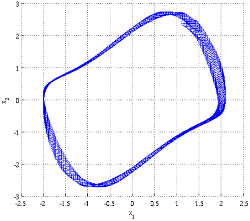


Brusselator

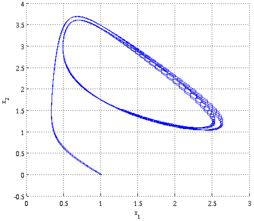


Rössler attractor

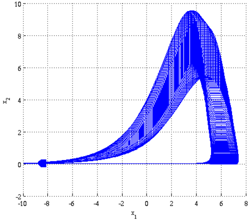
Continuous systems:



Van-der-Pol oscillator



Brusselator



Rössler attractor

How can we apply Taylor models to hybrid systems?

Taylor model flowpipe construction for hybrid systems

- Time evolution:
  - Verified integration using Taylor models
  - Compute flowpipe/invariant intersections
- For a discrete transition:
  - Compute flowpipe/guard intersections
  - Compute the image of the reset mapping

Over-approximate an intersection

We use three techniques in combination to over-approximate the intersection $(p, I) \cap G$:

1. Domain contraction
2. Range over-approximation
3. Template method

Intersection of a Taylor model (p, I) over domain $D \times T$ and a guard G:

$$\underbrace{\mathbf{x} = p(\mathbf{x}_0, t) + \mathbf{y} \ \wedge \ \mathbf{y} \in I \ \wedge \ \underbrace{\mathbf{x}_0 \in D \wedge t \in T}_{\text{Taylor model domain}}}_{\text{Taylor model}} \ \wedge \ \underbrace{G(\mathbf{x})}_{\text{guard predicate}}$$

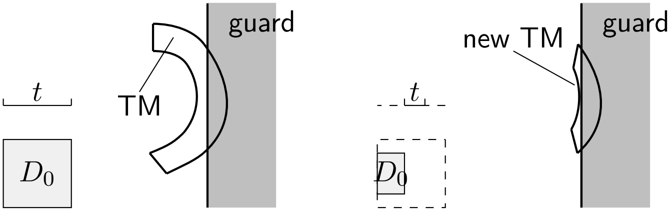
Intersection of a Taylor model (p, I) over domain $D \times T$ and a guard G:

$$\underbrace{\mathbf{x} = p(\mathbf{x}_0, t) + \mathbf{y} \;\wedge\; \mathbf{y} \in I \;\wedge\; \underbrace{\mathbf{x}_0 \in D \wedge t \in T}_{\text{Taylor model domain}}}_{\text{Taylor model}} \;\wedge\; \underbrace{G(\mathbf{x})}_{\text{guard predicate}}$$

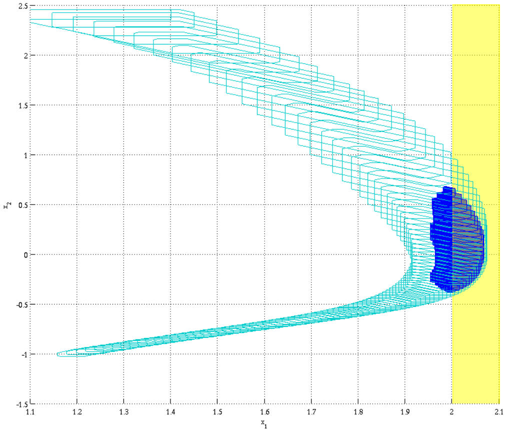
Domain contaction:

1. Split the domain in one dimension into two halves
2. Contract dimensions using interval constraint propagation
3. Drop empty (unsatisfiable) domain halves

1. Domain contraction: Example



1. Domain contraction: Example



2. Range over-approximation

1. Over-approximate (p, I) by a set S which can be a
   - support function
     Conservative over-approximation of a polynomial
   - zonotope
     Zonotopes are order 1 *Taylor models and vise versa*
2. Over-approximate $S \cap G$ by a Taylor model

[Sankaranarayanan et el., 2008], [Le Guernic et al., 2009]

Goal: Find a Taylor model $(p^*, 0)$ over domain

$$D_u = [\ell_1, u_1] \times [\ell_2, u_2] \times \cdots \times [\ell_n, u_n]$$

containing the intersection $(p, I) \cap G$

[Sankaranarayanan et al., 2008], [Gulwani, 2008]

3. Template method

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Tasks:

- Fix p^*
- Compute proper parameters $\ell_1, \ldots, \ell_n, u_1, \ldots, u_n$

[Sankaranarayanan et al., 2008], [Gulwani, 2008]

3. Template method

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containing the intersection $(p, I) \cap G$

Tasks:

- Fix p^*
- Compute proper parameters $\ell_1, \ldots, \ell_n, u_1, \ldots, u_n$

$$\forall \mathbf{x}.((\mathbf{x} = p(\mathbf{y}) + \mathbf{z} \wedge \mathbf{y} \in D \wedge \mathbf{z} \in I \wedge \mathbf{x} \in G) \\ \rightarrow \exists \mathbf{x}_0.(\mathbf{x} = p^*(\mathbf{x}_0) \wedge \mathbf{x}_0 \in D_u))$$

[Sankaranarayanan et al., 2008], [Gulwani, 2008]

Example: Glycemic control in diabetic patients

G	plasma glucose concentration above the basal value G_B
I	plasma insulin concentration above the basal value I_B
X	insulin concentration in an interstitial chamber

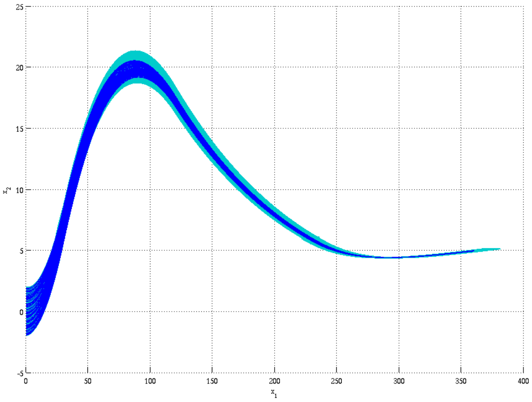
$$\begin{aligned} \frac{dG}{dt} &= -p_1 G - X(G + G_B) + g(t) \\ \frac{dX}{dt} &= -p_2 X + p_3 I \\ \frac{dI}{dt} &= -n(I + I_b) + \frac{1}{V_I} i(t) \end{aligned}$$

$g(t)$	infusion of glucose into the bloodstream
$i(t)$	infusion of insulin into the bloodstream

$$g(t) = \begin{cases} \frac{t}{60} & t \leq 30 \\ \frac{120-t}{180} & t \in [30, 120] \\ 0 & t \geq 120 \end{cases}$$

$$i(t) = \begin{cases} 1 + \frac{2G(t)}{9} & G(t) < 6 \\ \frac{50}{3} & G(t) \geq 6 \end{cases}$$

Glycemic Control in Diabetic Patients

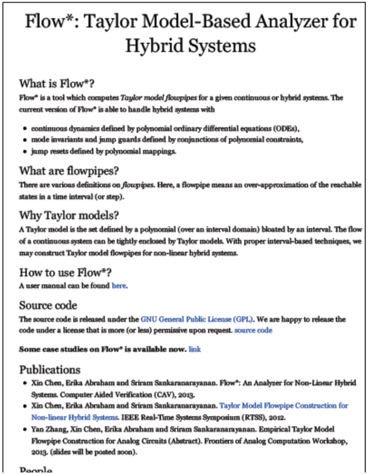


Order of the Taylor models: 9 Time step: 0.02 Time horizon: [0,360]

Total time: 1804 s Time of intersection: 443 s Memory: 410 MB

Fetch the tool

http://systems.cs.colorado.edu/research/cyberphysical/taylormodels/



Flow*: Taylor Model-Based Analyzer for Hybrid Systems

What is Flow*?

Flow* is a tool which computes *Taylor model flowpipes* for a given continuous or hybrid systems. The current version of Flow* is able to handle hybrid systems with

- continuous dynamics defined by polynomial ordinary differential equations (ODEs),
- mode invariants and jump guards defined by conjunctions of polynomial constraints,
- jump resets defined by polynomial mappings.

What are flowpipes?

There are various definitions on *flowpipes*. Here, a flowpipe means an over-approximation of the reachable states in a time interval (or step).

Why Taylor models?

A Taylor model is the set defined by a polynomial (over an interval domain) bloated by an interval. The flow of a continuous system can be tightly enclosed by Taylor models. With proper interval-based techniques, we may construct Taylor model flowpipes for non-linear hybrid systems.

How to use Flow*?

A user manual can be found here.

Source code

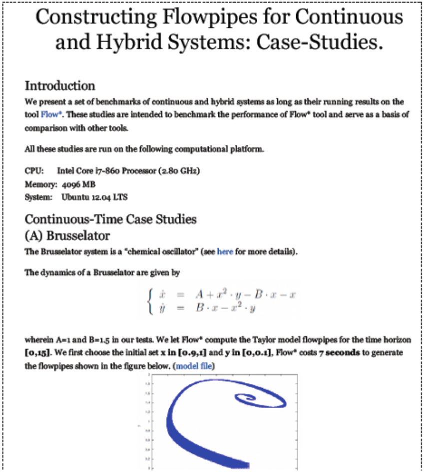
The source code is released under the GNU General Public License (GPL). We are happy to release the code under a license that is more (or less) permissive upon request. source code

Some case studies on Flow* is available now. link

Publications

- Xin Chen, Erika Abraham and Sriram Sankaranarayanan. Flow*: An Analyzer for Non-Linear Hybrid Systems. Computer Aided Verification (CAV), 2013.
- Xin Chen, Erika Abraham and Sriram Sankaranarayanan. Taylor Model Flowpipe Construction for Non-linear Hybrid Systems. IEEE Real-Time Systems Symposium (RTSS), 2012.
- Yan Zhang, Xin Chen, Erika Abraham and Sriram Sankaranarayanan. Empirical Taylor Model Flowpipe Construction for Analog Circuits (Abstract). Frontiers of Analog Computation Workshop, 2013. (slides will be posted soon).

People



Constructing Flowpipes for Continuous and Hybrid Systems: Case-Studies.

Introduction

We present a set of benchmarks of continuous and hybrid systems as long as their running results on the tool Flow*. These studies are intended to benchmark the performance of Flow* tool and serve as a basis of comparison with other tools.

All these studies are run on the following computational platform.

CPU: Intel Core i7-860 Processor (2.80 GHz)
Memory: 4096 MB
System: Ubuntu 12.04 LTS

Continuous-Time Case Studies

(A) Brusselator

The Brusselator system is a "chemical oscillator" (see here for more details).

The dynamics of a Brusselator are given by

$$\begin{cases} \dot{x} = A + x^2 \cdot y - B \cdot x - x \\ \dot{y} = B \cdot x - x^2 \cdot y \end{cases}$$

wherein A=1 and B=1.5 in our tests. We let Flow* compute the Taylor model flowpipes for the time horizon **[0,15]**. We first choose the initial set **x in [0.9,1]** and **y in [0,0.1]**, Flow* costs **7 seconds** to generate the flowpipes shown in the figure below. (model file)