Modeling and Analysis of Hybrid Systems
Convex polyhedra

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Informatik 2 - Theory of Hybrid Systems
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SS 2015
Polyhedra
Convex polyhedra

Definition

A polyhedron in $\mathbb{R}^d$ is the solution set to a finite number of linear inequalities with real coefficients in $d$ real variables. A bounded polyhedron is called polytope.
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$$\forall x, y \in S. \forall \lambda \in [0, 1] \subseteq \mathbb{R}. \lambda x + (1 - \lambda)y \in S.$$ 

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Polyhedra are convex sets.
Depending on the form of the representation we distinguish between

- $\mathcal{H}$-polytopes and
- $\mathcal{V}$-polytopes
Intersection of a finite set of halfspaces

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Definition (Closed halfspace)

A $d$-dimensional \textbf{closed halfspace} is a set $\mathcal{H} = \{ x \in \mathbb{R}^d \mid c^T x \leq z \}$ for some $c \in \mathbb{R}^d$, called the \textbf{normal} of the halfspace, and a $z \in \mathbb{R}$. 
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\textbf{Definition ($\mathcal{H}$-polyhedron, $\mathcal{H}$-polytope)}

A $d$-dimensional $\mathcal{H}$-\textbf{polyhedron} $P = \bigcap_{i=1}^{n} \mathcal{H}_i$ is the intersection of finitely many closed halfspaces. A bounded $\mathcal{H}$-polyhedron is called an $\mathcal{H}$-\textbf{polytope}.

The facets of a $d$-dimensional $\mathcal{H}$-polytope are $d-1$-dimensional $\mathcal{H}$-polytopes.
An $\mathcal{H}$-polytope

$$P = \bigcap_{i=1}^{n} \mathcal{H}_i = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^d \mid c_i \cdot x \leq z_i\}$$

can also be written in the form

$$P = \{x \in \mathbb{R}^d \mid Cx \leq z\}.$$  

We call $(C, z)$ the $\mathcal{H}$-representation of the polytope.
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- Each row of $C$ is the normal vector to the $i$th facet of the polytope.
**H-polytopes**

An $H$-polytope

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We call $(C, z)$ the $H$-representation of the polytope.

- Each row of $C$ is the normal vector to the $i$th facet of the polytope.
- An $H$-polytope $P$ has a finite number of vertices $V(P)$. 
Convex hull of a finite set of points
Convex hull of a finite set of points
**Definition (Convex hull)**

Given a set $V \subseteq \mathbb{R}^d$, the **convex hull** $\text{conv}(V)$ of $V$ is the smallest convex set that contains $V$.

**Definition ($\mathcal{V}$-polytope)**

A $\mathcal{V}$-polytope $P = \text{conv}(V)$ is the convex hull of a finite set $V \subset \mathbb{R}^d$. We call $V$ the $\mathcal{V}$-representation of the polytope. Note that all $\mathcal{V}$-polytopes are bounded. Unbounded polyhedra can be represented by extending convex hulls with conical hulls.
\( \mathcal{V} \)-polytopes

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For a finite set \( V = \{v_1, \ldots, v_n\} \), its convex hull can be computed by

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\text{conv}(V) = \{ x \in \mathbb{R}^d \mid \exists \lambda_1, \ldots, \lambda_n \in [0, 1] \subseteq \mathbb{R}. \sum_{i=1}^{n} \lambda_i = 1 \land \sum_{i=1}^{n} \lambda_i v_i = x \}. 
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Note that all $\mathcal{V}$-polytopes are bounded.
Unbounded polyhedra can be represented by extending convex hulls with conical hulls.
Conical hull of a finite set of points
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If $U = \{u_1, \ldots, u_n\}$ is a finite set of points in $\mathbb{R}^d$, the conical hull of $U$ is defined by

$$\text{cone}(U) = \{x \mid x = \sum_{i=1}^{n} \lambda_i u_i, \lambda_i \geq 0\}. \quad (1)$$
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Each polyhedra $P \subseteq \mathbb{R}^d$ can be represented by two finite sets $V, U \subseteq \mathbb{R}^d$ such that

$$P = conv(V) \oplus cone(U).$$

If $U$ is empty then $P$ is bounded (e.g., a polytope).
Motzkin’s theorem

- For each $\mathcal{H}$-polytope, the convex hull of its vertices defines the same set in the form of a $\mathcal{V}$-polytope, and vice versa,

- each set defined as a $\mathcal{V}$-polytope can be also given as an $\mathcal{H}$-polytope by computing the halfspaces defined by its facets.

The translations between the $\mathcal{H}$- and the $\mathcal{V}$-representations of polytopes can be exponential in the state space dimension $d$. 
If we represent reachable sets of hybrid automata by polytopes, we need some operations like

- membership computation,
- intersection, or the
- union of two polytopes.
Operations: Membership

Membership for $p \in \mathbb{R}^d$:
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- $\mathcal{H}$-polytope defined by $Cx \leq z$:

  - just substitute $p$ for $x$ to check if the inequation holds.

  - $V$-polytope defined by the vertex set $V$:
    - check satisfiability of
      $$\exists \lambda_1, \ldots, \lambda_n \in [0,1] \subseteq \mathbb{R}^d.$$ 
      $$\sum_{i=1}^{n} \lambda_i = 1 \land \sum_{i=1}^{n} \lambda_i v_i = x.$$

  - Alternatively:
    - convert the $V$-polytope into an $\mathcal{H}$-polytope by computing its facets.
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Alternatively: convert the \( \mathcal{V} \)-polytope into an \( \mathcal{H} \)-polytope by computing its facets.
Intersection for two polytopes $P_1$ and $P_2$:

- $\mathcal{H}$-polytopes defined by $C_1x \leq z_1$ and $C_2x \leq z_2$:
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- $\mathcal{H}$-polytopes defined by $C_1x \leq z_1$ and $C_2x \leq z_2$:
  the resulting $\mathcal{H}$-polytope is defined by $\left( \begin{array}{c} C_1 \\ C_2 \end{array} \right) x \leq \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right)$.

- $\mathcal{V}$-polytopes defined by $V_1$ and $V_2$: 

  Convert $P_1$ and $P_2$ to $\mathcal{H}$-polytopes and convert the result back to a $\mathcal{V}$-polytope.
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- $\mathcal{V}$-polytopes defined by $V_1$ and $V_2$:
  $\mathcal{V}$-representation $V_1 \cup V_2$.
- $\mathcal{H}$-polytopes defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$: 
Note that the union of two convex polytopes is in general not a convex polytope.

→ take the convex hull of the union.

- $\mathcal{V}$-polytopes defined by $V_1$ and $V_2$:
  \[ V_1 \cup V_2. \]

- $\mathcal{H}$-polytopes defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$:
  convert to $\mathcal{V}$-polytopes and compute back the result.
Hardness of the convex hull computation

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It could also be hard to translate a $\mathcal{V}$-polytope to an $\mathcal{H}$-polytope or vice versa.