Modeling and Analysis of Hybrid Systems
Timed automata

Prof. Dr. Erika Ábrahám

Informatik 2 - Theory of Hybrid Systems
RWTH Aachen University

SS 2015
Christel Baier and Joost-Pieter Katoen: Principles of Model Checking
Contents

1 Motivation

2 Timed automata

3 TCTL
Correctness in *time-critical* systems not only depends on the logical result of the computation but also on the time at which the results are produced.
Correctness in **time-critical** systems not only depends on the logical result of the computation but also on the time at which the results are produced.

Thus if we model such systems, we also need to model the time.
Correctness in time-critical systems not only depends on the logical result of the computation but also on the time at which the results are produced.

Thus if we model such systems, we also need to model the time. The first choice in modelling: discrete or continuous time?
Discrete-time systems

Advantages:
- Conceptually simple
- Each action lasts for a single time unit (tick)
- Action $\alpha$ lasts $k > 0$ time units; $k - 1$ ticks followed by $\alpha$

Disadvantages:
- Leads to large transition systems
- Minimal time between two actions is a multiple of the tick

Logic: CTL or LTL extended with syntactic sugar

- $X \varphi$: $\varphi$ holds after one tick
- $X^k \varphi$: $\varphi$ holds after $k$ ticks
- $F \leq k \varphi$: $\varphi$ occurs within $k$ ticks

We deal in this lecture with continuous-time models.
Discrete-time systems

Advantages:

- conceptually simple
- each action lasts for a single time unit (tick)
- action $\alpha$ lasts $k > 0$ time units $\leadsto k - 1$ ticks followed by $\alpha$
Discrete-time systems

Advantages:

- conceptually simple
- each action lasts for a single time unit (tick)
- action $\alpha$ lasts $k > 0$ time units $\leadsto k-1$ ticks followed by $\alpha$

Disadvantages:

- leads to large transition systems
- minimal time between two actions is a multiple of the tick

Logic: CTL or LTL extended with syntactic sugar

- $X \varphi$: $\varphi$ holds after one tick
- $X^k \varphi$: $\varphi$ holds after $k$ ticks
- $F^k \varphi$: $\varphi$ occurs within $k$ ticks

We deal in this lecture with continuous-time models.
Discrete-time systems

Advantages:

- conceptually simple
- each action lasts for a single time unit (tick)
- action \( \alpha \) lasts \( k > 0 \) time units \( \sim k - 1 \) ticks followed by \( \alpha \)

Disadvantages:

- leads to large transition systems
- minimal time between two actions is a multiple of the tick

Logic: CTL or LTL extended with syntactic sugar

- \( X \varphi \) : \( \varphi \) holds after one tick
- \( X^k \varphi \) : \( \varphi \) holds after \( k \) ticks
- \( F^k \varphi \) : \( \varphi \) occurs within \( k \) ticks
Discrete-time systems

Advantages:

■ conceptually simple
■ each action lasts for a single time unit (tick)
■ action $\alpha$ lasts $k > 0$ time units $\sim k - 1$ ticks followed by $\alpha$

Disadvantages:

■ leads to large transition systems
■ minimal time between two actions is a multiple of the tick

Logic: CTL or LTL extended with syntactic sugar

$X\varphi$ : $\varphi$ holds after one tick

$X^k\varphi$ : $\varphi$ holds after $k$ ticks

$F^{\leq k}\varphi$ : $\varphi$ occurs within $k$ ticks

We deal in this lecture with continuous-time models.
Contents

1 Motivation

2 Timed automata

3 TCTL
Timed automata

- Measure time: finite set $C$ of clocks $x, y, z, \ldots$
- Clocks increase their value implicitly as time progresses
- All clocks proceed at rate 1
Timed automata

- Measure time: finite set $\mathcal{C}$ of clocks $x, y, z, \ldots$
- Clocks increase their value implicitly as time progresses
- All clocks proceed at rate 1
- Limited clock access

**Read access:**

Atomic clock constraints:

$$acc ::= x < c | x \leq c | x > c | x \geq c$$

with $c \in \mathbb{N}$ ($c \in \mathbb{Q}$) and $x \in \mathcal{C}$.

Clock constraints:

$$g ::= acc | g \land g$$

Syntactic sugar: $true$, $x \in [c_1, c_2)$, $c_1 \leq x < c_2$, $x = c, \ldots$

**ACC($\mathcal{C}$):** set of atomic clock constraints over $\mathcal{C}$

**CC($\mathcal{C}$):** set of clock constraints over $\mathcal{C}$

**Write access:** Clock reset sets clock value to 0
Semantics of clock constraints

Given a set $C$ of clocks, a clock valuation
Semantics of clock constraints

Given a set $C$ of clocks, a clock valuation $\nu: C \rightarrow \mathbb{R}_{\geq 0}$ assigns a non-negative value to each clock. We use $V_C$ to denote the set of clock valuations for the clock set $C$.

Definition (Semantics of clock constraints)

$\models \ x \leq c \iff \nu(x) \leq c$

$\models \leq V_C \times CC(C)$
Semantics of clock constraints

Given a set $C$ of clocks, a clock valuation $\nu : C \rightarrow \mathbb{R}_{\geq 0}$ assigns a non-negative value to each clock. We use $V_C$ to denote the set of clock valuations for the clock set $C$.

**Definition (Semantics of clock constraints)**

For a set $C$ of clocks, $x \in C$, $\nu \in V_C$, $c \in \mathbb{N}$, and $g, g' \in CC(C)$, let $\models \subseteq V_C \times CC(C)$ be defined by

- $\nu \models x < c$ iff $\nu(x) < c$
- $\nu \models x \leq c$ iff $\nu(x) \leq c$
- $\nu \models x > c$ iff $\nu(x) > c$
- $\nu \models x \geq c$ iff $\nu(x) \geq c$
- $\nu \models g \land g'$ iff $\nu \models g$ and $\nu \models g'$
### Semantics of clock access

#### Definition (Time delay, clock reset)

For a set $C$ of clocks, $\nu \in V_C$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in C$.

For a valuation $\nu \in V_C$ and a clock set $R \subseteq C$ we define $\text{reset}_R$ in $\nu$ to be the valuation resulting from $\nu$ by resetting all clocks from $R$:

$$(\text{reset}_R \in \nu)(y) = \begin{cases} \nu(x) & \text{if } x \not\in R \\ 0 & \text{else} \end{cases}$$

For a single clock $x \in C$ we write $\text{reset}_x \in \nu$. 

<table>
<thead>
<tr>
<th>valuation for $C = {x, y}$</th>
<th>value of $x$</th>
<th>value of $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$\nu + 9$</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>$\text{reset}_x \in (\nu + 9)$</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>$\text{reset}_{{x, y}} \in \nu$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Semantics of clock access

Definition (Time delay, clock reset)

For a set $C$ of clocks, $\nu \in V_C$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with

$$(\nu + c)(x) = \nu(x) + c$$
Semantics of clock access

Definition (Time delay, clock reset)

For a set $\mathcal{C}$ of clocks, $\nu \in V_{\mathcal{C}}$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in \mathcal{C}$. 
**Definition (Time delay, clock reset)**

- For a set $\mathcal{C}$ of clocks, $\nu \in V_\mathcal{C}$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in \mathcal{C}$.

- For a valuation $\nu \in V_\mathcal{C}$ and a clock set $R \subseteq \mathcal{C}$ we define $\text{reset } R \text{ in } \nu$ to be

$$
(\text{reset } R \text{ in } \nu)(x) = \begin{cases} 
\nu(x) & \text{if } x \not\in R \\
0 & \text{else}
\end{cases}
$$

For a single clock $x \in \mathcal{C}$ we write $\text{reset } x \text{ in } \nu$. 
Definition (Time delay, clock reset)

- For a set $C$ of clocks, $\nu \in V_C$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in C$.

- For a valuation $\nu \in V_C$ and a clock set $R \subseteq C$ we define $\text{reset } R \text{ in } \nu$ to be the valuation resulting from $\nu$ by resetting all clocks from $R$:

$$ (\text{reset } R \text{ in } \nu)(y) = \begin{cases} \nu(x) & \text{if } x \not\in R \\ 0 & \text{else} \end{cases} $$

For a single clock $x \in C$ we write $\text{reset } x \text{ in } \nu$. 
Semantics of clock access

Definition (Time delay, clock reset)

- For a set $C$ of clocks, $\nu \in V_C$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in C$.

- For a valuation $\nu \in V_C$ and a clock set $R \subseteq C$ we define $\text{reset } R \text{ in } \nu$ to be the valuation resulting from $\nu$ by resetting all clocks from $R$:
  \[
  (\text{reset } R \text{ in } \nu)(y) = \begin{cases} 
  \nu(x) & \text{if } x \notin R \\
  0 & \text{else}
  \end{cases}
  \]

For a single clock $x \in C$ we write $\text{reset } x \text{ in } \nu$.

<table>
<thead>
<tr>
<th>valuation for $C = {x, y}$</th>
<th>value of $x$</th>
<th>value of $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$\nu + 9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu + 9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{reset } x \text{ in } (\nu + 9)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(\text{reset } x \text{ in } \nu) + 9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{reset } {x, y} \text{ in } \nu$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Semantics of clock access

Definition (Time delay, clock reset)

- For a set $C$ of clocks, $\nu \in V_C$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in C$.

- For a valuation $\nu \in V_C$ and a clock set $R \subseteq C$ we define $\text{reset } R \text{ in } \nu$ to be the valuation resulting from $\nu$ by resetting all clocks from $R$:

  $$(\text{reset } R \text{ in } \nu)(y) = \begin{cases} 
  \nu(x) & \text{if } x \notin R \\
  0 & \text{else}
  \end{cases}$$

For a single clock $x \in C$ we write $\text{reset } x \text{ in } \nu$.

<table>
<thead>
<tr>
<th>valuation for $C = {x, y}$</th>
<th>value of $x$</th>
<th>value of $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$\nu + 9$</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>$\text{reset } x \text{ in } (\nu + 9)$</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>$(\text{reset } x \text{ in } \nu) + 9$</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>$\text{reset } {x, y} \text{ in } \nu$</td>
<td>14</td>
<td>10</td>
</tr>
</tbody>
</table>
Semantics of clock access

Definition (Time delay, clock reset)

- For a set $C$ of clocks, $\nu \in V_C$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in C$.

- For a valuation $\nu \in V_C$ and a clock set $R \subseteq C$ we define $\text{reset } R \text{ in } \nu$ to be the valuation resulting from $\nu$ by resetting all clocks from $R$:

$$ (\text{reset } R \text{ in } \nu)(y) = \begin{cases} \nu(x) & \text{if } x \notin R \\ 0 & \text{else} \end{cases} $$

For a single clock $x \in C$ we write $\text{reset } x \text{ in } \nu$.

<table>
<thead>
<tr>
<th>valuation for $C = {x, y}$</th>
<th>value of $x$</th>
<th>value of $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$\nu + 9$</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>$\text{reset } x \text{ in } (\nu + 9)$</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>$(\text{reset } x \text{ in } \nu) + 9$</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>$\text{reset } {x, y} \text{ in } \nu$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Semantics of clock access

Definition (Time delay, clock reset)

- For a set $\mathcal{C}$ of clocks, $\nu \in V_{\mathcal{C}}$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in \mathcal{C}$.

- For a valuation $\nu \in V_{\mathcal{C}}$ and a clock set $R \subseteq \mathcal{C}$ we define $\text{reset } R \text{ in } \nu$ to be the valuation resulting from $\nu$ by resetting all clocks from $R$:

$$
(\text{reset } R \text{ in } \nu)(y) = \begin{cases} 
\nu(x) & \text{if } x \notin R \\
0 & \text{else}
\end{cases}
$$

For a single clock $x \in \mathcal{C}$ we write $\text{reset } x \text{ in } \nu$.

<table>
<thead>
<tr>
<th>valuation for $\mathcal{C} = {x, y}$</th>
<th>value of $x$</th>
<th>value of $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$\nu + 9$</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>$\text{reset } x \text{ in } (\nu + 9)$</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>$(\text{reset } x \text{ in } \nu) + 9$</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>$\text{reset } {x, y} \text{ in } \nu$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Semantics of clock access

Definition (Time delay, clock reset)

- For a set $C$ of clocks, $\nu \in V_C$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in C$.

- For a valuation $\nu \in V_C$ and a clock set $R \subseteq C$ we define \textit{reset $R$ in $\nu$} to be the valuation resulting from $\nu$ by resetting all clocks from $R$:

$$
(reset R \text{ in } \nu)(y) = \begin{cases} 
\nu(x) & \text{if } x \notin R \\
0 & \text{else}
\end{cases}
$$

For a single clock $x \in C$ we write \textit{reset $x$ in $\nu$}.

<table>
<thead>
<tr>
<th>valuation for $C = {x, y}$</th>
<th>value of $x$</th>
<th>value of $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$\nu + 9$</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>$\text{reset } x \text{ in } (\nu + 9)$</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>$(\text{reset } x \text{ in } \nu) + 9$</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>$\text{reset } {x, y} \text{ in } \nu$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
A timed automaton is a special hybrid automaton:

- All variables are clocks.
- States $\sigma \in \Sigma$ are pairs of a location and a clock valuation $\langle l, v \rangle$.
- Edges are defined by
  - source and target locations,
  - a label,
  - a guard: clock constraint specifying enabling,
  - a set of clocks to be reset.
- Invariants are clock constraints $\mu \subseteq V_C \times V_C = \{(v, v') | v = g \land v' = \text{reset } R \in v\}$. 

$\mu \subseteq V_C \times V_C$
Timed automaton

Definition (Syntax of timed automata)

A timed automaton $\mathcal{T} = (\text{Loc}, \mathcal{C}, \text{Lab}, \text{Edge}, \text{Inv}, \text{Init})$ is a tuple with

- $\text{Loc}$ is a finite set of locations,
- $\mathcal{C}$ is a finite set of clocks,
- $\text{Lab}$ is a finite set of synchronisation labels,
- $\text{Edge} \subseteq \text{Loc} \times \text{Lab} \times (\text{CC}(\mathcal{C}) \times 2^\mathcal{C}) \times \text{Loc}$ is a finite set of edges,
- $\text{Inv} : \text{Loc} \to \text{CC}(\mathcal{C})$ is a function assigning an invariant to each location, and
- $\text{Init} \subseteq \Sigma$ with $\nu(x) = 0$ for all $x \in \mathcal{C}$ and all $(l, \nu) \in \text{Init}$.

We call the variables in $\mathcal{C}$ clocks. We also use the notation $\xymatrix{ l \ar@{->}[r]^a & l' \text{:} (g,R) }$ to state that there exists an edge $(l, a, (g, R), l') \in \text{Edge}$.

Note: (1) no explicit activities given (2) restricted logic for constraints
\[(g_1 \circ g_2)(x) = g_1(g_2(x)) = g_1 \circ g_2\]
Analogously to Kripke structures, we can additionally define

- a set of atomic propositions $AP$ and
- a labelling function $L : Loc \rightarrow 2^{AP}$

To model further system properties.
Operational semantics

\[ A_1 \quad A_2 \quad \ldots \quad A_n \quad \xrightarrow{\text{rule}} \quad (l, \nu \ldots \nu \nu(a) \ldots \nu) \]

\[ (l, \nu \ldots \nu \nu(a) \ldots \nu) \quad \xrightarrow{\text{rule}} \quad (l, \nu \ldots \nu \nu(\nu(a)) \ldots \nu) \]

\[ (l, \nu \ldots \nu \nu(a) \ldots \nu) \quad \xrightarrow{\text{reset}} \quad (l, \nu \ldots \nu \nu(a) \ldots \nu) \]

\[ (l, \nu \ldots \nu \nu(a) \ldots \nu) \quad \xrightarrow{\text{reset}} \quad (l, \nu \ldots \nu \nu(a) \ldots \nu) \]

\[ (l, \nu \ldots \nu \nu(a) \ldots \nu) \quad \xrightarrow{\text{reset}} \quad (l, \nu \ldots \nu \nu(a) \ldots \nu) \]

\[ v, v' = J_\nu(e) \quad \nu' = \nu + t \]

\[ (l, \nu) \quad \xrightarrow{\tau} \quad (l, \nu') \]

\[ (l, a : (q, R), l') \in \text{Edge} \quad \nu = q \]

\[ (l, \nu) \quad \xrightarrow{a} \quad (l', \nu') \]

Initial path: path \[ \sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots \]

with \[ \sigma_0 = (l_0, \nu_0) \]

and \[ \nu_0(x) = 0 \quad \text{for all} \quad x \in C \]

Reachability of a state: exists an initial path leading to the state.
Operational semantics

\[(l, a, (g, R), l') \in \text{Edge}\]

\[\nu \models g \quad \nu' = \text{reset } R \text{ in } \nu \quad \nu' \models \text{Inv}(l')\]

\[\frac{}{(l, \nu) \overset{a}{\rightarrow} (l', \nu')}\]

\[t > 0 \quad \nu' = \nu + t \quad \nu' \models \text{Inv}(l)\]

\[\frac{}{(l, \nu) \overset{t}{\rightarrow} (l, \nu')}\]
Operational semantics

\[(l, a, (g, R), l') \in \text{Edge} \]
\[
\nu \models g \quad \nu' = \text{reset } R \text{ in } \nu \quad \nu' \models \text{Inv}(l') \quad \text{Rule} \text{ Discrete}
\]
\[
(l, \nu) \xrightarrow{a} (l', \nu')
\]

\[t > 0 \quad \nu' = \nu + t \quad \nu' \models \text{Inv}(l) \quad \text{Rule} \text{ Time}
\]
\[
(l, \nu) \xrightarrow{t} (l, \nu')
\]

- **Execution step:** \[\xrightarrow{a} \cup \xrightarrow{t}\]
- **Path:** \[\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots \text{ with } \sigma_0 = (l_0, \nu_0) \text{ and } \nu_0 \in \text{Inv}(l_0)\]
- **Initial path:** path \[\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots \text{ with } \sigma_0 = (l_0, \nu_0), l_0 \in \text{Init} \text{ and } \nu_0(x) = 0 \text{ for all } x \in \mathcal{C}\]
- **Reachability** of a state: exists an initial path leading to the state
Example: Timed Automaton

$x \geq 2, \text{reset}(x)$
Example: Timed Automaton

\[ x \geq 2, \text{ reset}(x) \]

Diagram:
- State: \( q_1 \)
- Transition: \( x \geq 2 \)
- Reset condition: \( \text{reset}(x) \)
Example: Timed Automaton

\[ x \geq 2, \text{ reset}(x) \]

\[ q_2 \]

\[ x \leq 3 \]
Example: Timed Automaton

\[ x \geq 2, \ reset(x) \]

\[ q_2 \]
\[ x \leq 3 \]
Example: Timed Automaton

$2 \leq x \leq 3$, \textit{reset}(x)
Example: Railroad Crossing

- **far**
  - **approach**
  - **near**
  - **enter**
  - **past**

- **up**
  - **lower**
  - **coming down**
  - **raise**

- **going up**
  - **raise**

- **down**

- **0**
  - **approach**
  - **raise**

- **1**
  - **lower**

- **3**
  - **exit**

- **2**

- **z ≤ 1**
  - **reset** (z)

- **x ≤ 1**
  - **reset** (x)

- **x ≥ 1**
  - **reset** (x)

- **y ≤ 0**
  - **reset** (y)

- **y > 2**
  - **exit**

- **y ≤ 5**

- **z ≤ 1**
  - **reset** (z)

- **z = 1**
  - **lower**

- **z ≥ 1**
  - **reset** (z)
Example: Railroad Crossing

- **far**
  - $y > 2$
  - enter
  - reset($y$)
  - approach

- **near**
  - $y \leq 5$
  - $y > 2$
  - enter
  - reset($y$)
  - approach

- **past**
  - $y \leq 5$
  - exit

- **up**
  - reset($x$)
  - lower
  - $x > 1$

- **coming down**
  - reset($x$)
  - $x \leq 1$

- **going up**
  - reset($x$)
  - lower
  - $x \leq 2$

- **down**
  - reset($x$)
  - raise

- **0**
  - reset($z$)
  - approach
  - $z \leq 1$

- **1**
  - raise
  - lower
  - $z = 1$

- **3**
  - reset($z$)
  - exit
  - lower
  - $z = 1$
Time divergence, timelock, and Zenoness

Paradox: Achilles and the tortoise

(Achilles was the great Greek hero of Homer’s The Iliad.)

“In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point where the pursued started, so that the slower must always hold a lead.”

—Aristotle, Physics VI:9, 239b15

- Not all paths of a timed automata represent realistic behaviour.
- Three essential phenomena: time convergence, timelock, Zenoness.

Zeno of Elea (ca.490 BC-ca.430 BC)

Aristotle (384 BC-322 BC)
Time convergence

Definition

For a timed automaton $\mathcal{T} = (\text{Loc}, C, \text{Lab}, \text{Edge}, \text{Inv}, \text{Init})$. we define $\text{ExecTime} : (\text{Lab} \cup \mathbb{R}^{\geq 0}) \rightarrow \mathbb{R}^{\geq 0}$ with

- $\text{ExecTime}(a) = 0$ for $a \in \text{Lab}$ and
- $\text{ExecTime}(d) = d$ for $d \in \mathbb{R}^{\geq 0}$.

Furthermore, for $\rho = s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} s_2 \xrightarrow{\alpha_2} \ldots$ we define

$$\text{ExecTime}(\rho) = \sum_{i=0}^{\infty} \text{ExecTime}(\alpha_i).$$

A path is time-divergent iff $\text{ExecTime}(\rho) = \infty$, and time-convergent otherwise.

- Time-convergent paths are not realistic, and are not considered in the semantics.
- Note: their existence cannot be avoided (in general).
Timelock

**Definition**

For a state $\sigma \in \Sigma$ let $Paths_{div}(\sigma)$ be the set of time-divergent paths starting in $\sigma$.
A state $\sigma \in \Sigma$ contains a timelock iff $Paths_{div}(\sigma) = \emptyset$.
A timed automaton is timelock-free iff none of its reachable states contains a timelock.

Timelocks are modelling flows and should be avoided.
Zenoness

Definition

An infinite path fragment $\pi$ is Zeno iff it is time-convergent and infinitely many discrete actions are executed within $\pi$. A timed automaton is non-Zeno iff no Zeno path starts in an initial state.

- Zeno paths represent non-realisable behaviour, since their execution would require infinitely fast processors.
- Though Zeno paths are modelling flows, they are not always easy to avoid.
- To check whether a timed automaton is non-Zeno is algorithmically difficult.
- Instead, sufficient conditions are considered that are simple to check, e.g., by static analysis.
Checking non-Zenoness

Theorem (Sufficient condition for non-Zenoness)

Let $T$ be a timed automaton with clocks $C$ such that for every control cycle

$$l_0 \xrightarrow{a_1:g_1,R_1} l_1 \xrightarrow{a_2:g_2,R_2} l_2 \cdots \xrightarrow{a_n:g_n,R_n} l_n = l_0$$

in $T$ there exists a clock $x \in C$ such that

- $x \in R_i$ for some $0 < i \leq n$, and
- for all evaluations $\nu \in V$ there exist some $0 < j \leq n$ and $d \in \mathbb{N}^>0$ with

$$\nu(x) < d \quad \text{implies} \quad (\nu \not\models Inv(l_j) \text{ or } \nu \not\models g_j).$$

Then $T$ is non-Zeno.
TCTL

How to describe the behaviour of timed automata?

Logic: TCTL, a real-time variant of CTL

Syntax:

State formulae

\[ \psi ::= \text{true} \mid a \mid \neg \psi \mid \exists \phi \mid \forall \phi \mid \psi \land \psi \mid \psi \lor \psi \]

Path formulae:

\[ \varphi ::= \psi U J \psi \mid \neg \varphi \mid \exists^3 \psi \mid \forall^3 \psi \]

with \( J \subseteq \mathbb{R}^{\geq 0} \) is an interval with integer bounds (open right bound may be \( \infty \)).

Note: no next-time operator
TCTL syntax

Syntactic sugar:

\[
\begin{align*}
\mathcal{F}^J \psi & := \text{true} \mathcal{U}^J \psi \\
\mathcal{E} \mathcal{G}^J \psi & := \neg \mathcal{A} \mathcal{F}^J \neg \psi \\
\mathcal{A} \mathcal{G}^J \psi & := \neg \mathcal{E} \mathcal{F}^J \neg \psi \\
\psi_1 \mathcal{U} \psi_1 & := \psi_1 \mathcal{U}^{[0,\infty)} \psi_2 \\
\mathcal{F} \psi & := \mathcal{F}^{[0,\infty)} \psi \\
\mathcal{G} \psi & := \mathcal{G}^{[0,\infty)} \psi
\end{align*}
\]
TCTL semantics

Definition (TCTL continuous semantics)

Let $\mathcal{T} = (Loc, C, Lab, Edge, Inv, Init)$ be a timed automaton, $AP$ a set of atomic propositions, and $L : Loc \rightarrow 2^{AP}$ a state labelling function. The function $\models$ assigns a truth value to each TCTL state and path formulae as follows:

$\models \subseteq (T \times 2) \times TCTL$  

<table>
<thead>
<tr>
<th>Formula</th>
<th>$\models$ Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\top$</td>
<td>$\sigma \models true$</td>
</tr>
<tr>
<td>$\sigma \models a$</td>
<td>$a \in L(\sigma) = L(\ell)$</td>
</tr>
<tr>
<td>$\sigma \models g$</td>
<td>$\sigma \models g \iff \forall (g) = true$</td>
</tr>
<tr>
<td>$\sigma \models \neg \psi$</td>
<td>$\sigma \not\models \psi$</td>
</tr>
<tr>
<td>$\sigma \models \psi_1 \land \psi_2$</td>
<td>$\sigma \models \psi_1$ and $\sigma \models \psi_2$</td>
</tr>
<tr>
<td>$\sigma \models E\varphi$</td>
<td>$\pi \models \varphi$ for some $\pi \in Paths_{div}(\sigma)$</td>
</tr>
<tr>
<td>$\sigma \models A\varphi$</td>
<td>$\pi \models \varphi$ for all $\pi \in Paths_{div}(\sigma)$</td>
</tr>
</tbody>
</table>

where $\sigma \in \Sigma$, $a \in AP$, $g \in ACC(C)$, $\psi$, $\psi_1$ and $\psi_2$ are TCTL state formulae, and $\varphi$ is a TCTL path formula.
TCTL semantics

Meaning of $U$: a time-divergent path satisfies $\psi_1 U^J \psi_2$ whenever at some time point in $J$ property $\psi_2$ holds and at all previous time instants $\psi_1$ is satisfied.
TCTL semantics

Definition (TCTL continuous semantics)

For a time-divergent path \( \pi = (\ell_0, \nu_0) \xrightarrow{\alpha_0} (\ell_1, \nu_1) \xrightarrow{\alpha_1} \ldots \) we define
\( \pi \models \psi_1 U^J \psi_2 \) iff

- \( \exists i \geq 0. (\ell_i, \nu_i + d) \models \psi_2 \) for some \( d \in [0, d_i] \) with
  \[
  \left( \sum_{k=0}^{i-1} d_k \right) + d \in J, \text{ and } \\
  \left( \sum_{k=0}^{i-1} d_k \right) + d \leq \left( \sum_{k=0}^{j-1} d_k \right) + d' \\
  \text{for any } d' \in [0, d_j] \text{ with}
  \]

where \( d_i = \text{ExecTime}(\alpha_i) \).
Definition

For a timed automaton $\mathcal{T}$ with clocks $C$ and locations $Loc$, and a TCTL state formula $\psi$ the satisfaction set $Sat(\psi)$ is defined by

$$Sat(\psi) = \{ s \in \Sigma \mid s \models \psi \}.$$  

$\mathcal{T}$ satisfies $\psi$ iff $\psi$ holds in all initial states:

$$\mathcal{T} \models \psi \iff \forall l_0 \in Init. (l_0, \nu_0) \models \psi$$

where $\nu_0(x) = 0$ for all $x \in C$. 

Satisfaction set
TCTL vs. CTL

- TCTL formulae with intervals $[0, \infty)$ may be considered as CTL formulae
- However, there is a difference due to time-convergent paths
- TCTL ranges over time-divergent paths, whereas CTL over all paths!
T:  
\[ x = 0 \rightarrow R \quad x \leq 3 \]  \[ x \geq 2 \rightarrow R \quad x \leq 1 \]  \[ x \geq 1 \rightarrow G \quad x \leq 20 \]  \[ x = 0 \rightarrow Y \]  

\[ T \models A (R \lor RY) \cup \leq 4 G \quad \checkmark \]
\[ T \models A \neg Y \quad \checkmark \]
\[ T \models AGA \neg Y \quad \checkmark \]
\[ T \models AF x \geq 1 \]
\[ T \models AGAF = 1 \quad true \]
\[ AGEF = 1 \quad true \]

\[ x = 0 \quad \Rightarrow x = \frac{1}{2} \quad \Rightarrow x = \frac{3}{4} \quad \Rightarrow \ldots \]

\[ \nu' = J_{\omega}(e) \quad \nu' = \nu + t \quad t > 0 \]
\[ (l, \nu) \xrightarrow{t} (l, \nu') \]

R