Modeling and Analysis of Hybrid Systems

Convex polyhedra

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Informatik 2 - Theory of Hybrid Systems
RWTH Aachen University

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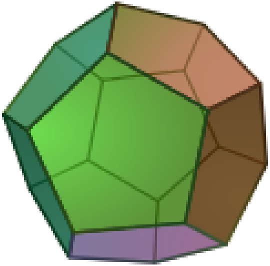
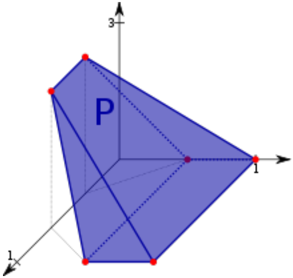
Contents

Polyhedra





Convex polyhedra

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A polyhedron in $\mathbb{R}^d$ is the solution set to a finite number of linear inequalities with ~~real~~ rational coefficients in d real variables. A bounded polyhedron is called polytope.

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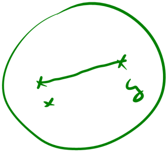
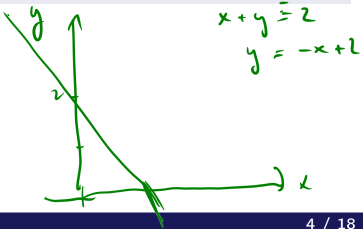
A set S is called convex, if

$$\forall x, y \in S.\ \forall \lambda \in [0,1] \subseteq \mathbb{R}.\ \lambda x + (1-\lambda)y \in S.$$

Polyhedra are convex sets.





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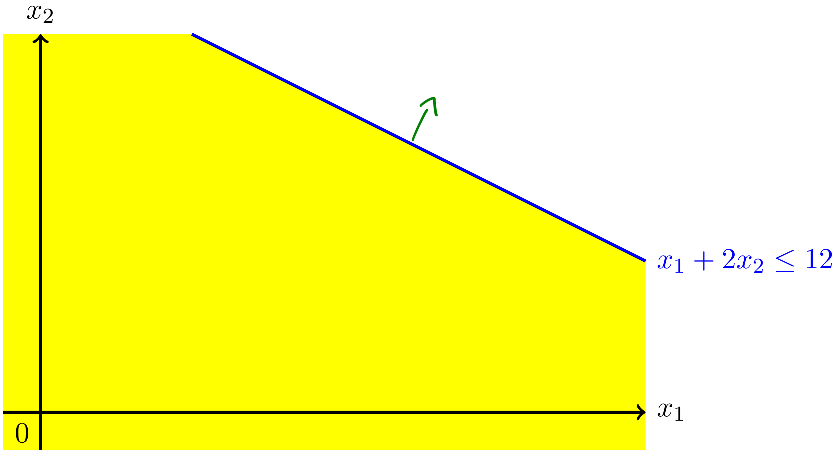
Polyhedra are convex sets.

Depending on the form of the representation we distinguish between

- $\mathcal{H}$-polytopes and
- $\mathcal{V}$-polytopes

Intersection of a finite set of halfspaces



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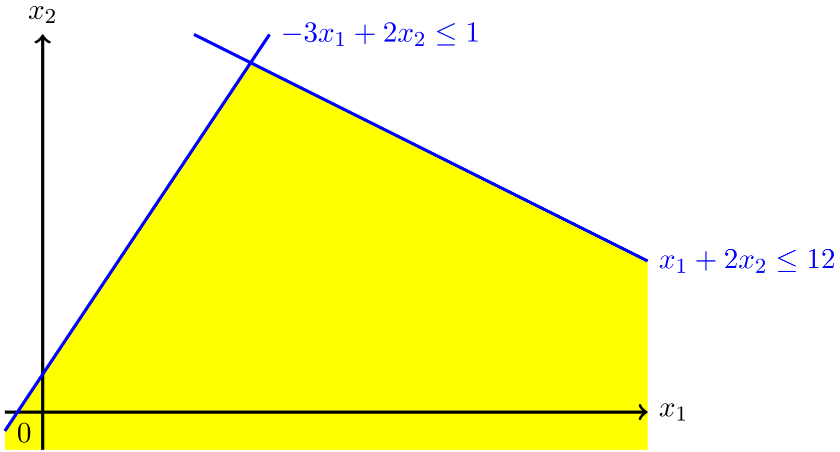

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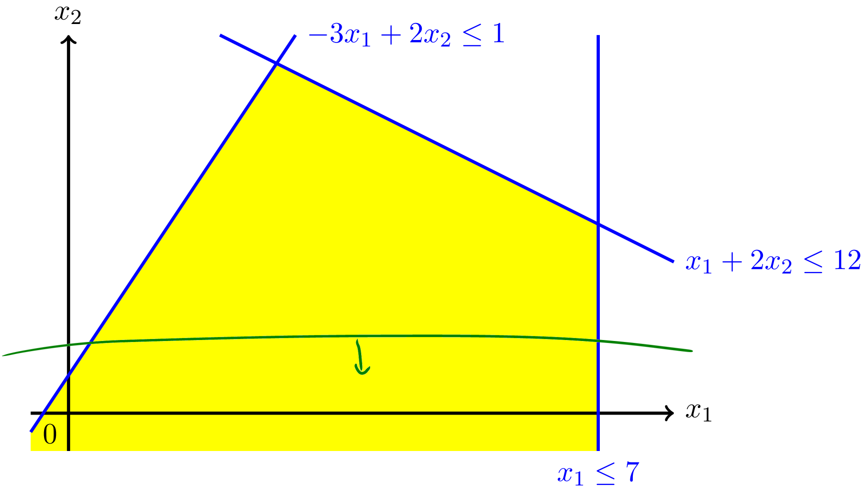

$\mathcal{H}$-polytopes

Definition (Closed halfspace)

A d-dimensional closed halfspace is a set $\mathcal{H} = \{x \in \mathbb{R}^d \mid c^T x \leq z\}$ for some $c \in \mathbb{R}^d$, called the normal of the halfspace, and a $z \in \mathbb{R}$.

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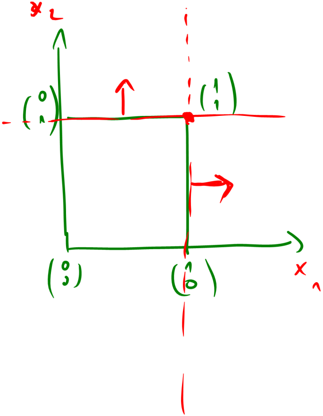
Definition ($\mathcal{H}$-polyhedron, $\mathcal{H}$-polytope)

A d-dimensional $\mathcal{H}$-polyhedron $P = \bigcap_{i=1}^{n} \mathcal{H}_i$ is the intersection of finitely many closed halfspaces. A bounded $\mathcal{H}$-polyhedron is called an $\mathcal{H}$-polytope.

The facets of a d-dimensional $\mathcal{H}$-polytope are $d-1$-dimensional $\mathcal{H}$-polytopes.

$$\left.\begin{matrix} c_1^T x \leq z_1 \\ \vdots \\ c_n^T x \leq z_n \end{matrix}\right\} \quad [C]\, x \leq [z]$$



$x_1 \leq 1$

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \leq \begin{pmatrix}1\\0\end{pmatrix}$$

$$C \cdot x \leq z$$

$\mathcal{H}$-polytopes

An $\mathcal{H}$-polytope

$$P = \bigcap_{i=1}^{n} \mathcal{H}_i = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^d \mid c_i \cdot x \leq z_i\}$$

can also be written in the form

$$P = \{x \in \mathbb{R}^d \mid Cx \leq z\}.$$

We call (C, z) the $\mathcal{H}$-representation of the polytope.

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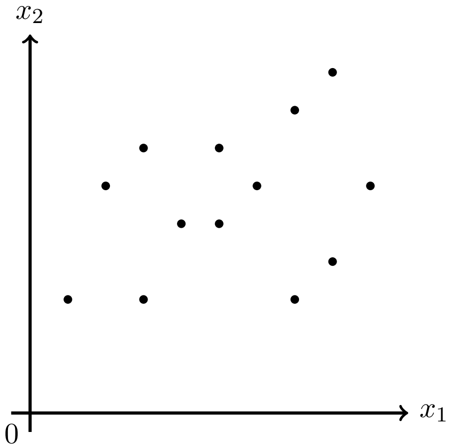
- An $\mathcal{H}$-polytope P has a finite number of vertices $V(P)$.

Convex hull of a finite set of points



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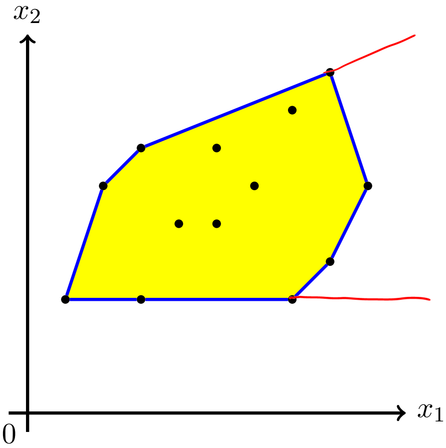

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For a finite set $V = \{v_1, \ldots, v_n\}$, its convex hull can be computed by

$$conv(V) = \{x \in \mathbb{R}^d \mid \exists \lambda_1, \ldots, \lambda_n \in [0,1] \subseteq \mathbb{R}.\ \sum_{i=1}^{n} \lambda_i = 1 \wedge \sum_{i=1}^{n} \lambda_i v_i = x\}.$$

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Definition ($\mathcal{V}$-polytope)

A $\mathcal{V}$-polytope $P = conv(V)$ is the convex hull of a finite set $V \subset \mathbb{R}^d$. We call V the $\mathcal{V}$-representation of the polytope.

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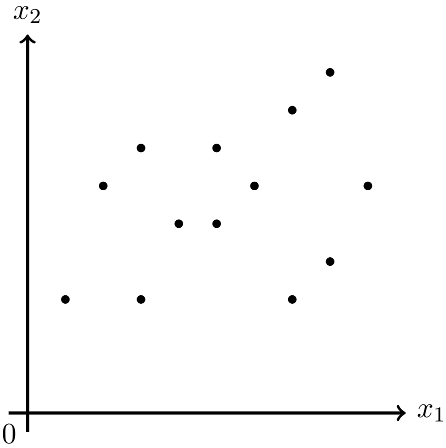
Unbounded polyhedra can be represented by extending convex hulls with conical hulls.

Conical hull of a finite set of points



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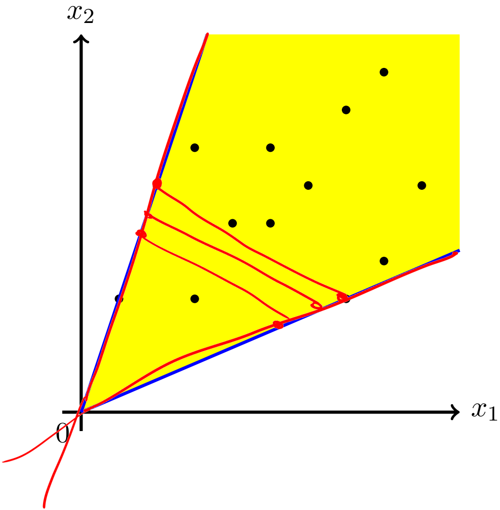

Conical hull of a finite set of points

If $U = \{u_1, \ldots, u_n\}$ is a finite set of points in $\mathbb{R}^d$, the conical hull of U is defined by

$$cone(U) = \{x \mid x = \sum_{i=1}^{n} \lambda_i u_i, \lambda_i \geq 0\}. \quad (1)$$

Conical hull of a finite set of points

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Each polyhedra $P \subseteq \mathbb{R}^d$ can be represented by two finite sets $V, U \subseteq \mathbb{R}^d$ such that

$$P = conv(V) \oplus cone(U) \ .$$

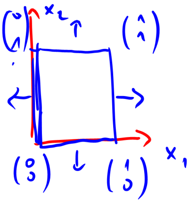
If U is empty then P is bounded (e.g., a polytope).

Motzkin's theorem

- For each $\mathcal{H}$-polytope, the convex hull of its vertices defines the same set in the form of a $\mathcal{V}$-polytope, and vice versa,
- each set defined as a $\mathcal{V}$-polytope can be also given as an $\mathcal{H}$-polytope by computing the halfspaces defined by its facets.

The translations between the $\mathcal{H}$- and the $\mathcal{V}$-representations of polytopes can be exponential in the state space dimension d.

Contents



$$H: \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{z}$$

$$C \cdot x \leq z$$

$$(C, z)$$

$$V: \left\{ \binom{0}{0}, \binom{1}{0}, \binom{0}{1}, \binom{1}{1} \right\}$$

$$\exists \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \ \bigwedge_{i=1}^{4} 0 \leq \lambda_i \leq 1 \ \wedge \ \sum_{i=1}^{4} \lambda_i = 1 \ \wedge \ \lambda_1 \binom{0}{0} + \lambda_2 \binom{1}{0} + \lambda_3 \binom{0}{1} + \lambda_4 \binom{1}{1} = \binom{x_1}{x_2}$$

$$\lambda_1 = 1 - \lambda_2 - \lambda_3 - \lambda_4 \quad \begin{vmatrix} \lambda_2 + \lambda_4 = x_1 \\ \lambda_3 + \lambda_4 = x_2 \end{vmatrix} \quad \begin{matrix} \lambda_2 = x_1 - \lambda_4 \\ \lambda_3 = x_2 - \lambda_4 \end{matrix}$$

$$0 \leq 1 - x_1 + \lambda_4 - x_2 + \cancel{\lambda_4} - \cancel{\lambda_4} \leq 1$$

$$0 \leq x_1 - \lambda_4 \leq 1$$

$$0 \leq x_2 - \lambda_4 \leq 1$$

$$0 \leq \lambda_4 \leq 1$$

$$x_1 + x_2 - 1 \leq \lambda_4 \qquad \lambda_4 \leq x_1 + x_2$$

$$\lambda_4 \leq x_1 \qquad x_1 - 1 \leq \lambda_4$$

$$\lambda_4 \leq x_2 \qquad x_2 - 1 \leq \lambda_4 \qquad 0 \leq \lambda_4 \leq 1$$

$$x_2 \leq 1$$

$$x_1 \leq 1$$

$$0 \leq x_1$$

$$0 \leq x_2$$

$$[0,1] \times [0,1]$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 \leq 1$$
$$x_2 \leq 1$$
$$0 \leq x_1$$
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Operations

If we represent reachable sets of hybrid automata by polytopes, we need some operations like

- membership computation,
- intersection, or the
- union of two polytopes.

Operations: Membership

Membership for $p \in \mathbb{R}^d$:

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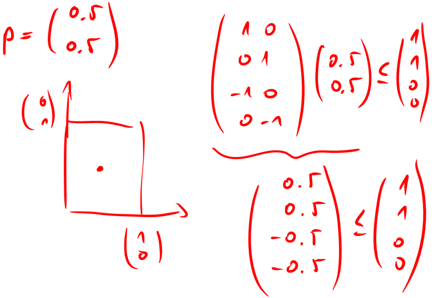
- $\mathcal{H}$-polytope defined by $Cx \leq z$:



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Alternatively:

Operations: Membership

Membership for $p \in \mathbb{R}^d$:

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$$\exists \lambda_1, \ldots, \lambda_n \in [0,1] \subseteq \mathbb{R}^d. \sum_{i=1}^{n} \lambda_i = 1 \wedge \sum_{i=1}^{n} \lambda_i v_i = x \; .$$

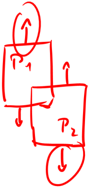
Alternatively: convert the $\mathcal{V}$-polytope into an $\mathcal{H}$-polytope by computing its facets.

Intersection

Intersection for two polytopes P_1 and P_2:

- $\mathcal{H}$-polytopes defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$:

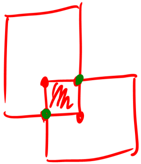
Intersection



Intersection for two polytopes P_1 and P_2:

- $\mathcal{H}$-polytopes defined by $C_1x \leq z_1$ and $C_2x \leq z_2$:
  the resulting $\mathcal{H}$-polytope is defined by $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} x \leq \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.
- $\mathcal{V}$-polytopes defined by V_1 and V_2:

Intersection



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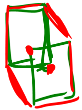
- $\mathcal{V}$-polytopes defined by V_1 and V_2:
  Convert P_1 and P_2 to $\mathcal{H}$-polytopes and convert the result back to a $\mathcal{V}$-polytope.

Union

Note that the union of two convex polytopes is in general not a convex polytope.

Union



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- $\mathcal{V}$-polytopes defined by V_1 and V_2:
  $\mathcal{V}$-representation $V_1 \cup V_2$.
- $\mathcal{H}$-polytopes defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$:

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- $\mathcal{V}$-polytopes defined by V_1 and V_2:
  $\mathcal{V}$-representation $V_1 \cup V_2$.
- $\mathcal{H}$-polytopes defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$:
  convert to $\mathcal{V}$-polytopes and compute back the result.

Hardness of the convex hull computation

	$\mapsto$	$conv$	$\oplus$	$\cap$
$\mathcal{V}$-polytope	easy	—	—	—
$\mathcal{H}$-polytope	hard	—	—	—
$\mathcal{V}$-polytope and $\mathcal{V}$-polytope	—	easy	easy	hard
$\mathcal{H}$-polytope and $\mathcal{H}$-polytope	—	hard	hard	easy
$\mathcal{V}$-polytope and $\mathcal{H}$-polytope	—	hard	hard	hard

It could also be hard to translate a $\mathcal{V}$-polytope to an $\mathcal{H}$-polytope or vice versa.