Practical Course: SMT Solving

Preprocessing, Union Find and Theory Propagation

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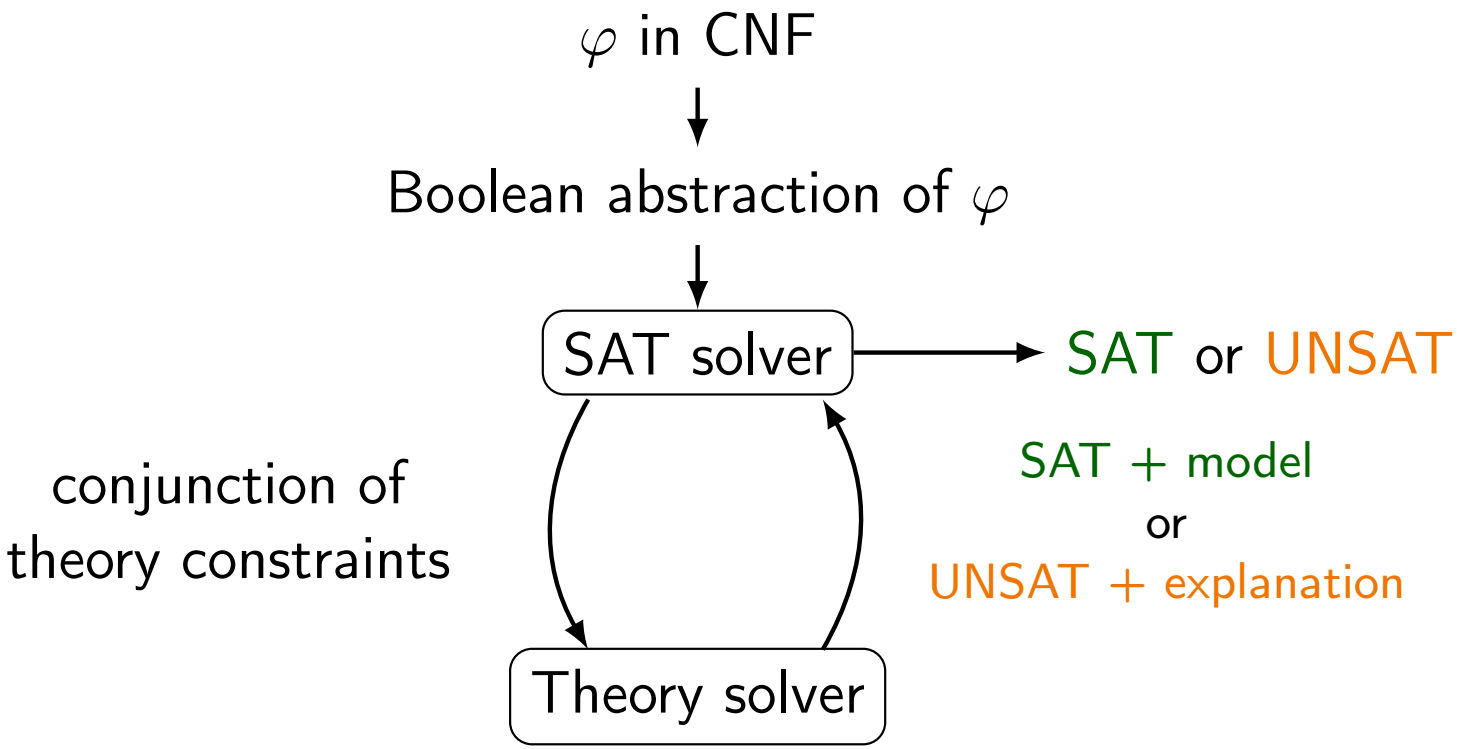
Winter term 2018/19

Preprocessing



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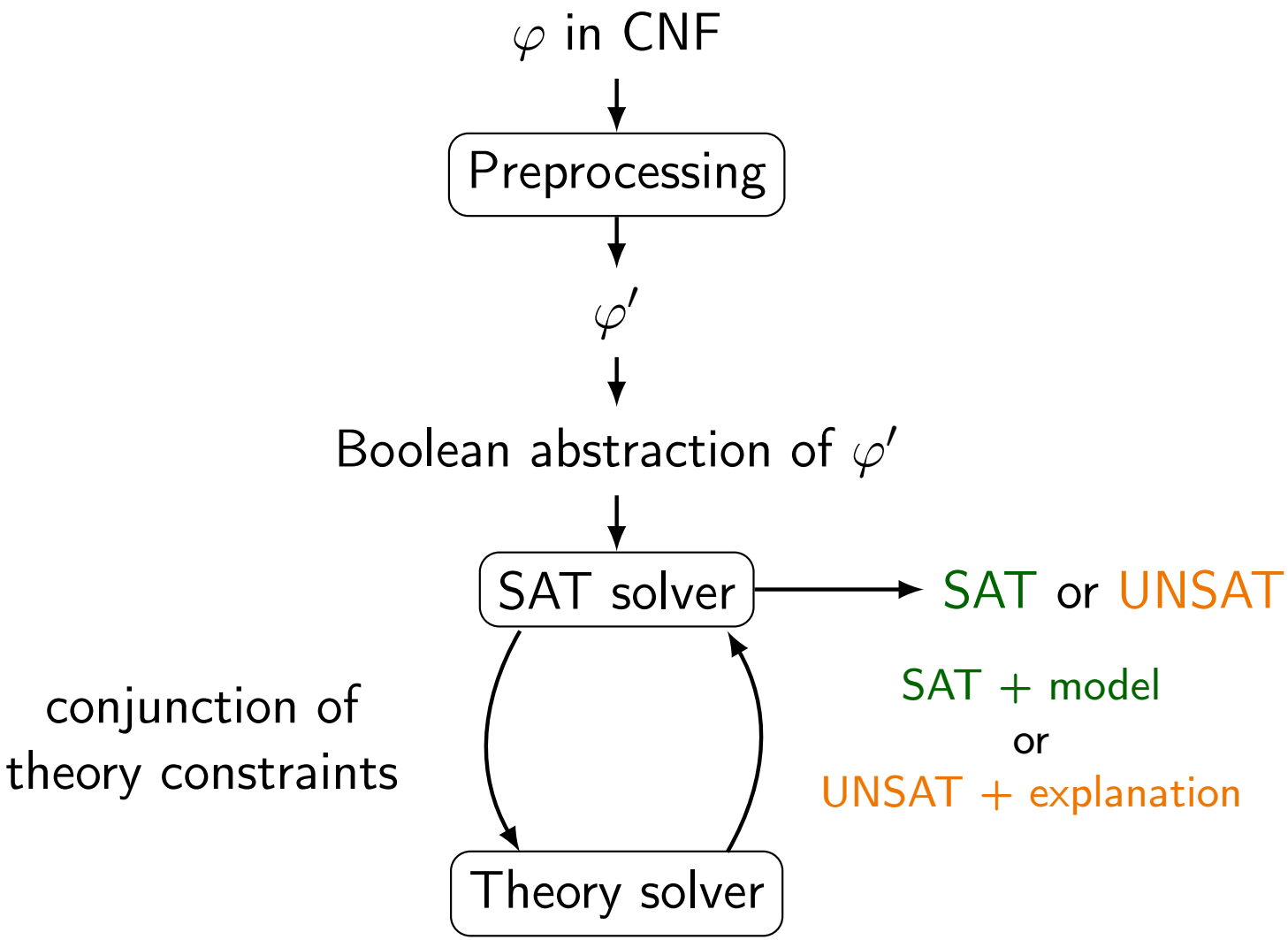

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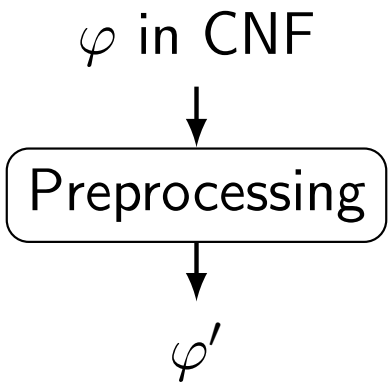

- equivalence preserving simplifications: $\varphi \equiv \varphi'$, e.g.

$$f(a) = b \ \wedge \ g(d, f(a)) = c \ \wedge \ h(f(a), f(a), f(a)) = e$$
$$\equiv f(a) = b \ \wedge \ g(d, b) = c \ \wedge \ h(b, b, b) = e$$

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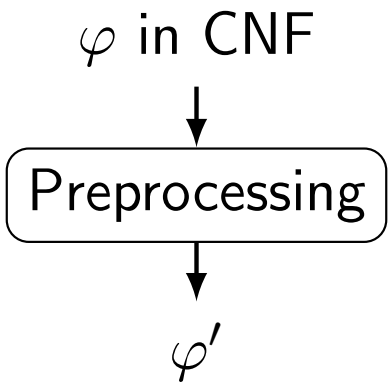

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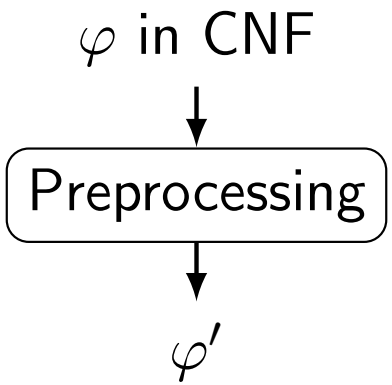

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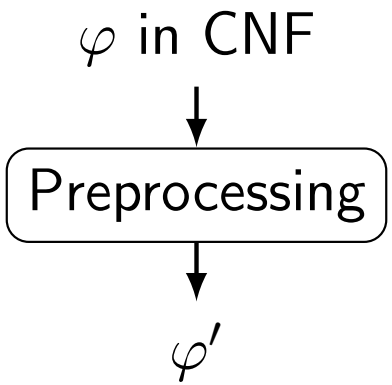

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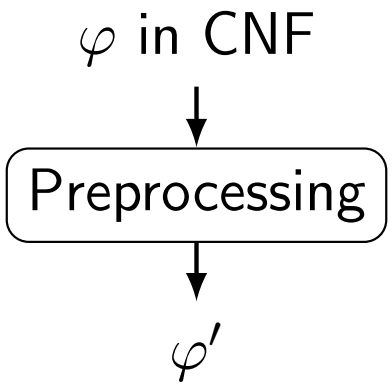

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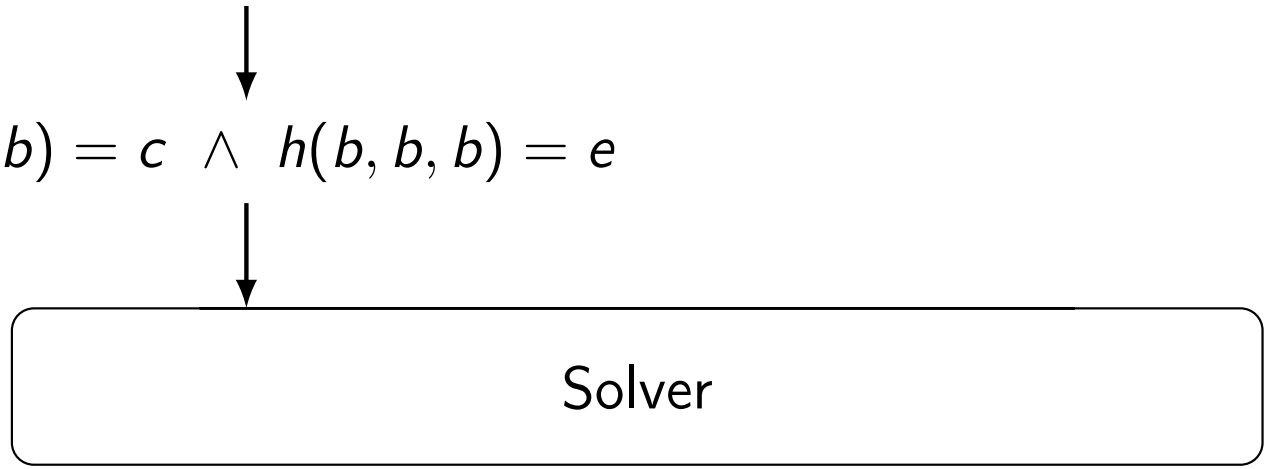

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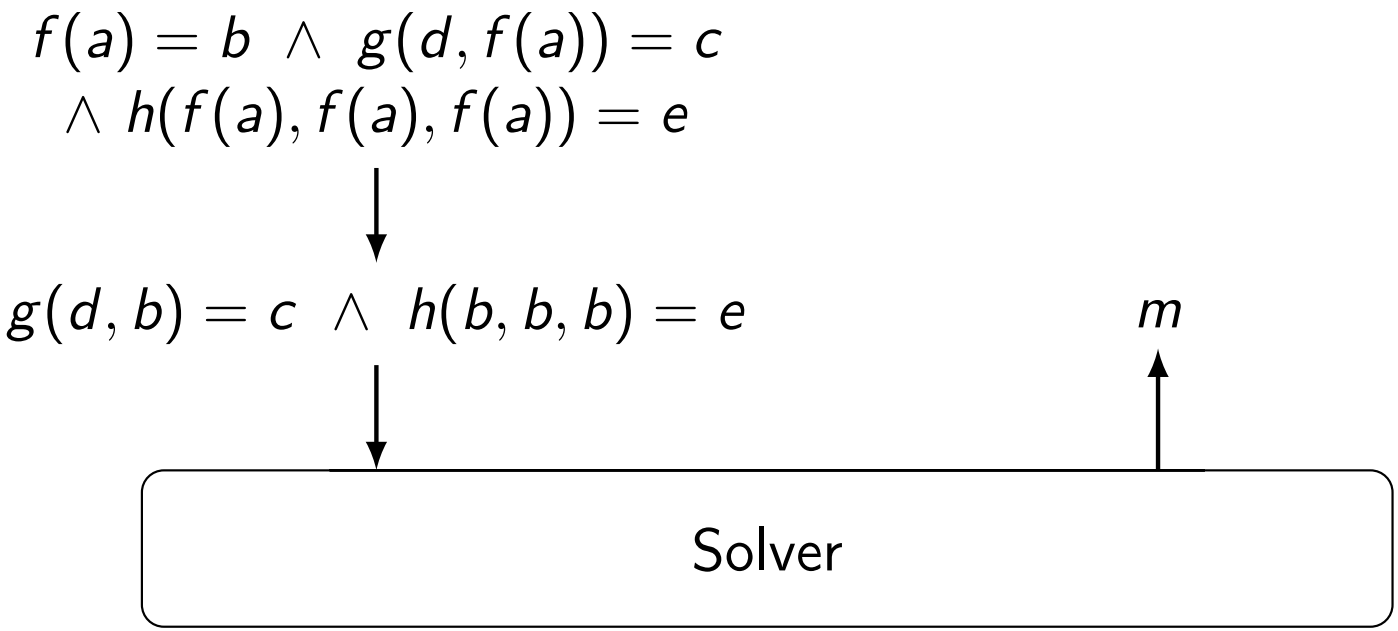

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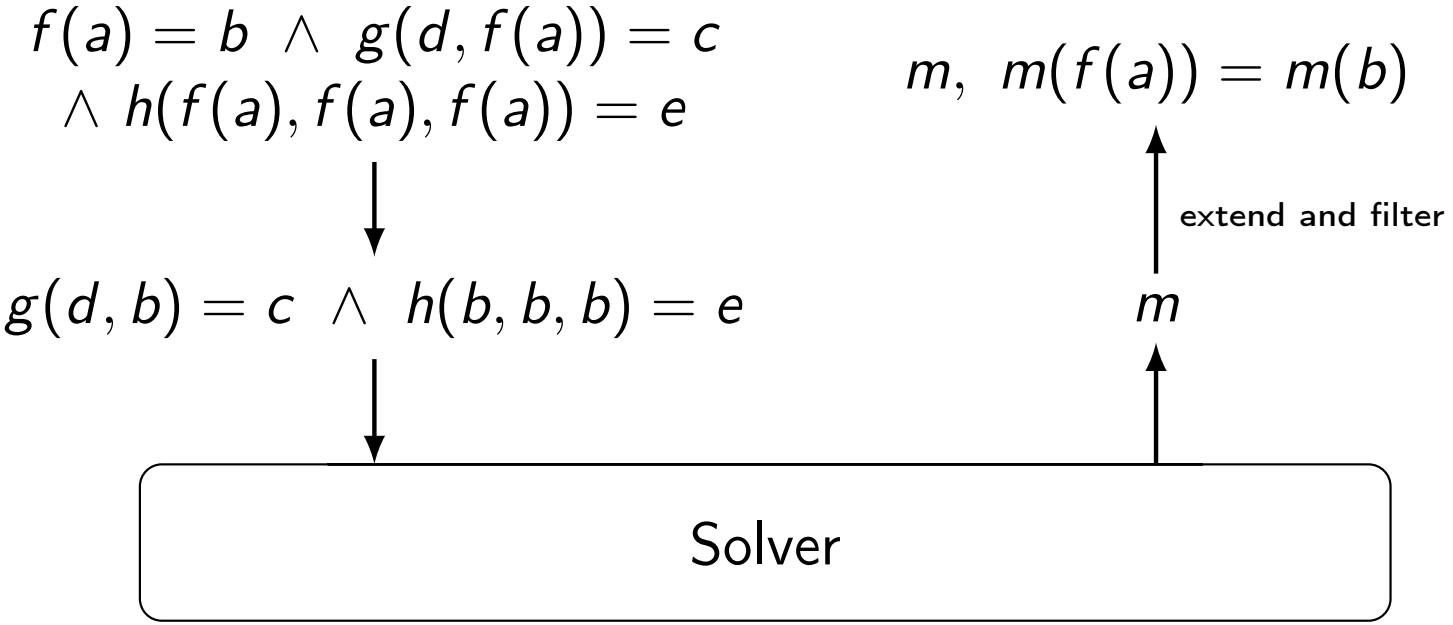

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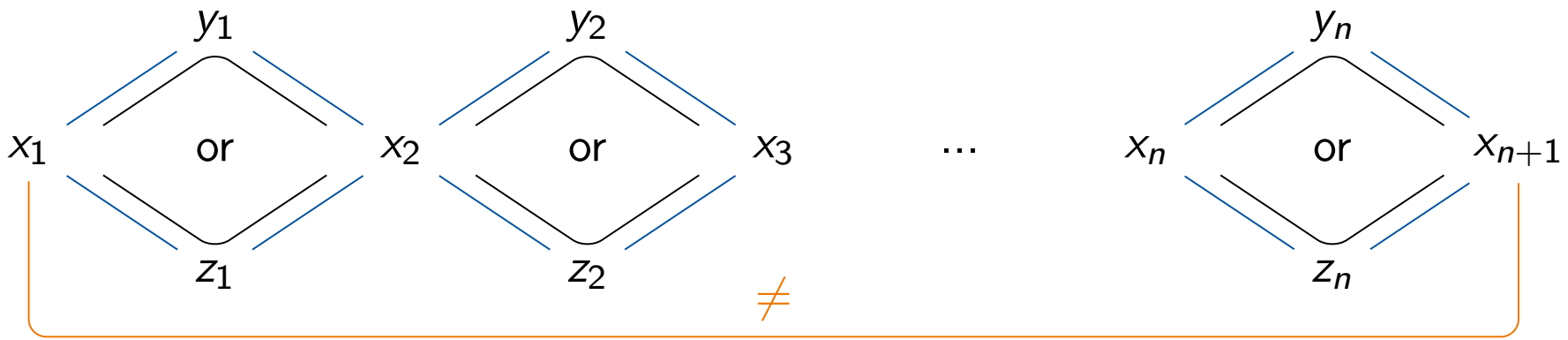
- $\varphi \ \mapsto \ \varphi[t/x]$ if t is a function of terms, for each subterm s of t occurs no literal $s = u$ (for any term u) in φ and x is a new variable

Diamond Chains



$$(x_1 = y_1 = x_2 \vee x_1 = z_1 = x_2) \wedge (x_2 = y_2 = x_3 \vee x_2 = z_2 = x_3) \wedge \cdots \wedge (x_n = y_n = x_{n+1} \vee x_n = z_n = x_{n+1})$$

$$\wedge\ x_1 \neq x_{n+1}$$

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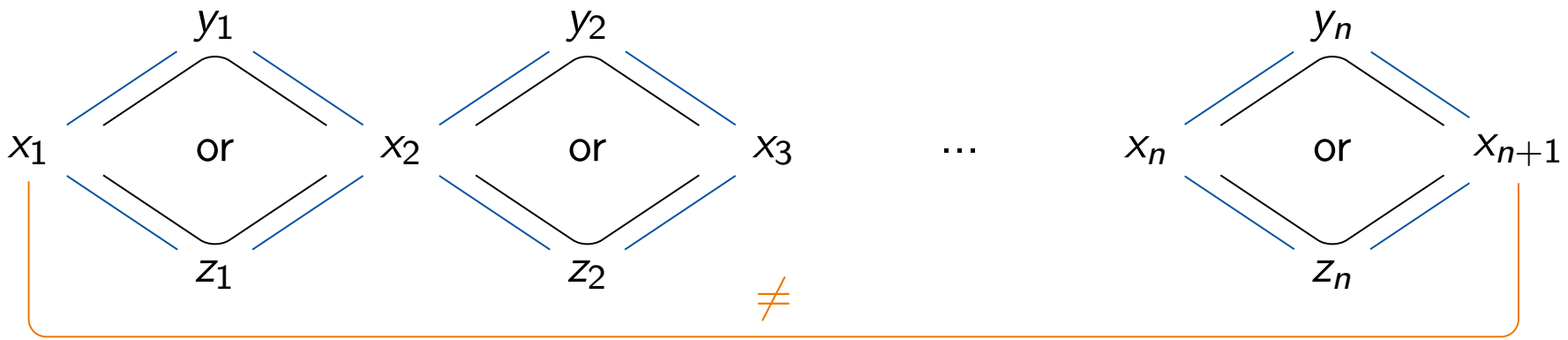

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Can be simplified to *false*. (But how?)

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Flatten:

- replace subterms with new variables
- e.g. $f(f(f(h, a), f(g, b)), c)$
  $\mapsto f(h, a) = d \wedge f(g, b) = e \wedge f(d, e)) = k \wedge f(k, c) = m$

Union Find [1]

- merge tree data structure F to store equivalence classes

[1] as in http://fm.csl.sri.com/SSFT12/smt-euf-arithmetic.pdf

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- x and y are equal if they are in the same tree (i.e. if $F^*(x) = F^*(y)$)
- let $sz(F, x)$ be the size of the tree containing x then merging the equivalence classes of x and y is defined as

$$union(F, x, y) = \begin{cases} F, & \textit{if } F^*(x) = F^*(y) \\ F[F^*(x) \mapsto F^*(y)], & \textit{if } sz(F, x) < sz(F, y) \\ F[F^*(y) \mapsto F^*(x)], & \textit{else} \end{cases}$$

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$$F = \{x_1 \mapsto x_1,\ x_2 \mapsto x_2,\ x_3 \mapsto x_3,\ x_4 \mapsto x_4,\ x_5 \mapsto x_5\}$$

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$\rightarrow$ x_2 and x_4 are in different classes

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$\rightarrow$ merge classes of x_4 and x_5; x_2 and x_4 are still in different classes

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- − non-incremental, has to be recomputed after backtracking

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- trade-off between costs of theory propagation and the Boolean search space reduction it provides
- in practice: incomplete theory propagation