Satisfiability Checking

Gröbner Bases

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Informatik 2
LuFG Theory of Hybrid Systems

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Motivation

For simplicity, we restrict to equations.

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This is a well-studied problem in algebra!
One approach: Gröbner bases.

Common roots = Varieties

Let $P = \{p_1, ..., p_k\} \subseteq \mathbb{R}[x_1, ..., x_n]$ and $\mathcal{V}(P)$ be the set of common real roots:

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We can simplify polynomials, maintaining $\mathcal{V}$!

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  $b := a \mod b$

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Generalized version

$A \subset \mathbb{Z}$

do

  $A' := A$

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Optimizations

- Normalize polynomials,
- Remove polynomials if factors have been found,

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If this one has no roots, the polynomials have no common roots.

Formally:

Theorem (Hilbert's weak Nullstellensatz)

Let I be an ideal in $K[x_1, x_2, ..., x_n]$, then

$$1 \in I \Leftrightarrow \bigcap_{p \in I} roots_K(p) = \emptyset$$

Buchbergers algorithm *partially expands* I to check if $1 \in I$.

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Unfortunately, K must be *algebraically closed*, i.e. $K \neq \mathbb{R}$, $K = \mathbb{C}$.

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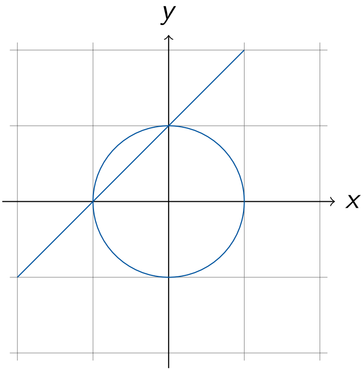
(If $\mathcal{V}(P)$ is finite, we can do this.)
$\Rightarrow$ We can determine unsatisfiability, not satisfiability.

Example - Graphical

$$x^2 + y^2 - 1 = 0 \wedge x - y + 1 = 0$$



Common roots: $(-1, 0)$, $(0, 1)$

Example - Algorithm

$$
\begin{aligned}
P = \{ & \\
& p_1 = x^2 + y^2 - 1, \\
& p_2 = x - y + 1 \\
& \}
\end{aligned}
$$

Example - Algorithm

$$p_1 \mod p_2 = 2y^2 - 2y$$

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\end{aligned}$$

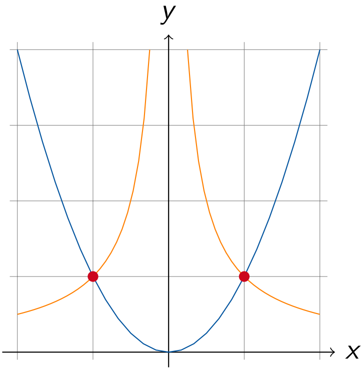
$1 \notin P$, hence no result in general.
With $y^2 - y = 0$ and $x - y + 1 = 0$ we can obtain $(-1, 0)$ and $(0, 1)$.

Example 1 – Finitely many common roots in $\mathbb{R}$

$$x^2 \cdot y^2 - 1 = 0 \wedge x^2 - y = 0$$

The Gröbner basis is:

$$P = \{x^2 - y, y^3 - 1\}$$



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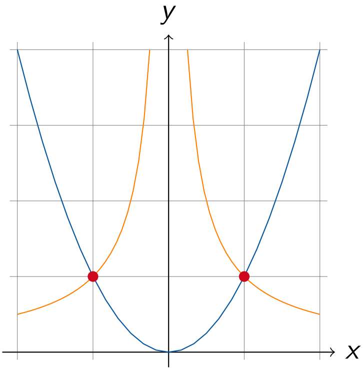
The Gröbner basis is:

$$P = \{x^2 - y, y^3 - 1\}$$

$1 \notin P$, hence there exists a solution in $\mathbb{C}$. The number of roots is finite, hence we can use the basis to obtain the solutions $(-1, 1)$ and $(1, 1)$ which are in $\mathbb{R}^2$.

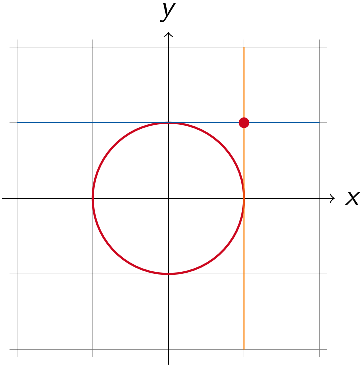


Example 2 – Infinitely many common roots in $\mathbb{R}$

$$x^2y - x^2 + y^3 - y^2 - y + 1 = 0 \wedge x^3 - x^2 + xy^2 - y - y^2 + 1 = 0$$

The Gröbner basis is:

$$P = \{p_1, p_2\}$$



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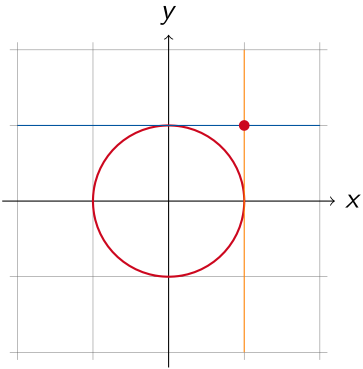
$1 \notin P$, hence there exists a solution in $\mathbb{C}^2$. The number of roots is infinite, hence there is no general way to obtain the actual solutions.



Example 3 – Common roots in $\mathbb{C}$

$x^2 + 1 = 0 \wedge x^3 + x = 0$

The Gröbner basis is:

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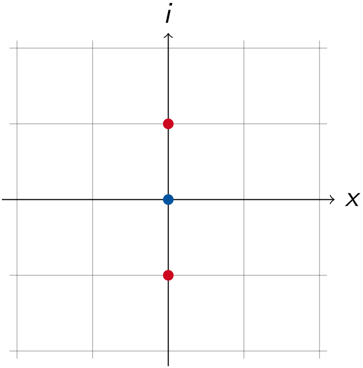

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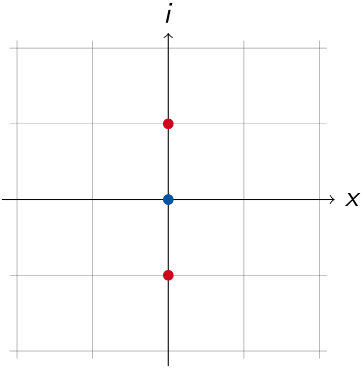
$1 \notin P$, hence there exists a solution, i.e. a common root, in $\mathbb{C}$. However, this solution is in $\mathbb{C} \setminus \mathbb{R}$.



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We only handled $=$. What about $\leq, \geq, <, >, \neq$?

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$$p(x) \leq 0 \Leftrightarrow p(x) + y^2 = 0$$
$$p(x) \geq 0 \Leftrightarrow p(x) - y^2 = 0$$
$$p(x) < 0 \Leftrightarrow p(x) \cdot y^2 + 1 = 0$$
$$p(x) > 0 \Leftrightarrow p(x) \cdot y^2 - 1 = 0$$
$$p(x) \neq 0 \Leftrightarrow p(x) \cdot y - 1 = 0$$

Extra: Ring

Let $(R, +, \cdot)$ be a ring, that is:

- $+$ is associative,
- $+$ is commutative,
- there is an additive identity 0,
- there is an additive inverse $-a$ for every $a \in R$,
- $\cdot$ is associative,
- there is a multiplicative identity 1,
- addition and multiplication distribute.

Examples:

- $\mathbb{Z}$ ($\cdot$ is commutative)
- $\mathbb{R}$ (is a field)
- $K[x_1, ..., x_n]$ (polynomial ring over field K)
- Our application: $\mathbb{R}[x_1, ..., x_n]$

Extra: Ideals

Ideal: A subset of a ring, that is *closed* under $+$ and *absorbs* $\cdot$.
Formally:

- $I \subseteq R$,
- $(I, +)$ is a subgroup of $(R, +)$, that is $0 \in I$, $-a \in I$ for all $a \in I$, $a + b \in I$ for all $a, b \in I$,
- $a \cdot b \in I$ for all $a \in I, b \in R$,
- $a \cdot b \in I$ for all $a \in R, b \in I$,

Examples:

- R is an ideal of R
- Even integers: $R = \mathbb{Z}$, $I = 2\mathbb{Z}$
- All polynomials divisible by $x^2 + 1$: $R = \mathbb{R}[x]$, $I = R/(x^2 + 1)$