Satisfiability Checking

Non-linear Real Arithmetic: Virtual Substitution

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Informatik 2
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Virtual substitution

- Is an existential quantifier elimination procedure:

$$\exists x_1.\ldots\exists x_n.\varphi' \quad \to \quad \exists x_1.\ldots\exists x_{n-1}.\psi',$$

  where $\varphi',\ \psi'$ quantifier free and $\exists x_1.\ldots\exists x_n.\varphi' \ \equiv\ \exists x_1.\ldots\exists x_{n-1}.\psi'$.

- Restricted in the degree of the variable to eliminate:

$$p(x) \sim 0 \text{ constraint of } \varphi \quad \Rightarrow \quad \text{degree of } x \text{ in } p(x) \text{ must be } \leq 2.$$

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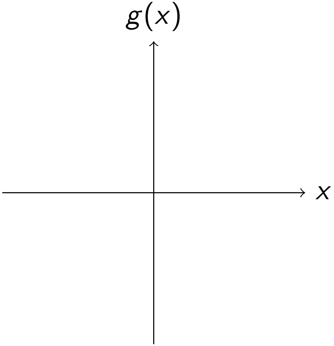
Virtual substitution constructs a finite set $T \subset \mathbb{R}$ of **test candidates** with

$$\exists x_1 \ldots \exists x_n . \ \varphi' \quad \equiv \quad \exists x_1 \ldots \exists x_{n-1} . \bigvee_{t \in T} \varphi'[t//x].$$

Construction of the set of test candidates T

$g(x) := ax^2 + bx + c$



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Given: A constraint $p \sim 0$, $p = ax^2 + bx + c$, $\sim \in \{=, <, >, \leq, \geq, \neq\}$.

Goal: Find out whether there is a satisfying solution (and return a solution or an explanation for unsatisfiability)

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Otherwise, the finite endpoints of p's sign-invariant regions are the zeros of p:

Linear in x :	$x_0 = -\frac{c}{b}$	, if	$a = 0 \quad \wedge \quad b \neq 0$
Quadratic in x, first solution:	$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$	, if	$a \neq 0 \quad \wedge \quad b^2 - 4ac \geq 0$
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Possible solution intervals for x in $p \sim 0$:

constraints	possible solution intervals ($0 \leq i,\ j \leq 2,\ i \neq j$)
$p = 0$	
$p < 0$ $p > 0$ $p \neq 0$	
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1. Zeros of the polynomial p
2. $-\infty$ (a “sufficiently small” value)

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Example: $xy + 1 < 0 \quad \rightarrow \quad$ 1. $-\frac{1}{y} + \epsilon$ if $y \neq 0$ $\qquad$ 2. $-\infty$

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\exists x.\varphi & \leftrightarrow & \varphi[-\infty//x] & & \lor \\
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& & (\varphi[-\frac{z}{y}//x] & \land(y \neq 0 \land z^2 > 0)) &
\end{array}
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1.	$-\infty$		(*from all constraints*)
2.	$\frac{1}{x}$	*if* $x \neq 0$	(*from* $xy - 1 = 0$)
3.	x		(*from* $y - x = 0$)
4.	$1 + \epsilon$		(*from* $y^2 - 1 < 0$)
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$$\begin{aligned}\exists x.\exists y.\varphi \quad \leftrightarrow \quad \exists x.\quad & (\varphi[-\infty//y]) && \vee \\ & (\varphi[\tfrac{1}{x}//y] \quad \wedge x \neq 0) && \vee \\ & (\varphi[x//y]) && \vee \\ & (\varphi[1 + \epsilon//y]) && \vee \\ & (\varphi[-1 + \epsilon//y]) &&\end{aligned}$$

Substituting test candidates T

Example: $\exists y . \exists x . \ (y = 0 \ \vee \ y^2 + 1 < 0) \ \wedge \ x - 3 \leq 0 \ \wedge \ xy + 1 < 0$

$$\overset{\text{eliminate x}}{\equiv}$$

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$$\ldots$$

Substitution of a variable by a test candidate in a constraint

- Standard substitution $\rightarrow$ expressions with ϵ, ∞, $\surd$ or division.
- Virtual substitution defines rules, which give an equivalent FO sentence over $(\mathbb{R}, +, \cdot, 0, 1, <)$ to the expression resulting by the above standard substitution.
- The substitution rules distinguish between
  - the constraint's relation symbol
  - the test candidate's type ($-\infty$, $+\epsilon$, contains $\surd$)

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Result:

1. Substitute x by $\frac{q+r\sqrt{t}}{s}$ in $g(x) = 0$ in the common way.
2. Transform the result to $\frac{\hat{q}+\hat{r}\sqrt{t}}{\hat{s}} = 0$ where $\hat{q}$, $\hat{r}$, and $\hat{s}$ are polynomials (always possible, proof exercise)
3. Compare:

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$$\Leftrightarrow \quad \hat{q}\hat{r} \leq 0 \ \wedge \ \|\hat{q}\| = \|\hat{r}\sqrt{t}\| \quad \Leftrightarrow \quad \hat{q}\hat{r} \leq 0 \wedge \hat{q}^2 - \hat{r}^2 t = 0$$

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$$\underbrace{g[e//x] < 0}_{\textbf{Case 1}} \vee \underbrace{g[e//x] = 0 \wedge g'[e//x] < 0}_{\textbf{Case 2}} \vee \underbrace{g[e//x] = 0 \wedge g'[e//x] = 0 \wedge g''[e//x] < 0}_{\textbf{Case 3}}$$

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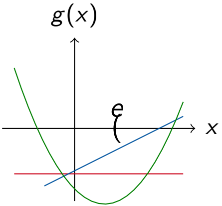
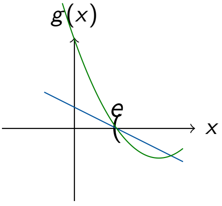
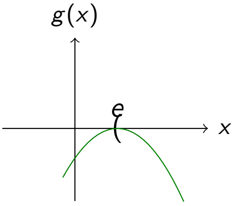
Explanation:



Case 1



Case 2



Case 3

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$\exists x.\exists y.\ ((\ xy - 1 = 0 \quad \vee \quad y - x \geq 0\ ) \quad \wedge \quad y^2 - 1 < 0\ )$

Eliminate y: 1. Test candidate: $-\infty$

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 & & \vee & (1 < 0 \ \vee \ (1 = 0 \ \wedge \ -x \geq 0))
\end{array}
$$

Virtual substitution: Example

$\exists x.\exists y.\ ((\ xy-1=0 \quad \vee \quad y-x\geq 0\ ) \quad \wedge \quad y^2-1<0\ )$

Eliminate y: 1. Test candidate: $-\infty$

$$
\begin{array}{lll}
\exists x.(& (& (xy-1=0)[-\infty//y] \\
 & \vee & (y-x\geq 0)[-\infty//y] \\
\wedge & & (y^2-1<0)[-\infty//y] \\
\Leftrightarrow \exists x.(& (& (x=0 \ \wedge \ -1=0) \\
 & \vee & (1<0 \ \vee \ (1=0 \ \wedge \ -x\geq 0)) \\
\wedge & & (1<0 \ \vee \ (1=0 \ \wedge \ 0>0) \ \vee \ (1=0 \ \wedge \ 0=0 \ \wedge \ -1<0))
\end{array}
$$

Virtual substitution: Example

$\exists x.\exists y.\ ((\ xy - 1 = 0 \quad \vee \quad y - x \geq 0\ ) \quad \wedge \quad y^2 - 1 < 0\ )$

Eliminate y: 1. Test candidate: $-\infty$

$$
\begin{array}{lll}
\exists x.(& (& (xy - 1 = 0)[-\infty//y] \\
& & \vee\ (y - x \geq 0)[-\infty//y] \\
& \wedge & (y^2 - 1 < 0)[-\infty//y] \\
\Leftrightarrow \exists x.(& (& (x = 0 \ \wedge \ -1 = 0) \\
& & \vee\ (1 < 0 \ \vee \ (1 = 0 \ \wedge \ -x \geq 0)) \\
& \wedge & (1 < 0 \ \vee \ (1 = 0 \ \wedge \ 0 > 0) \ \vee \ (1 = 0 \ \wedge \ 0 = 0 \ \wedge \ -1 < 0)) \\
\Leftrightarrow \exists x.(\mathit{false}) & &
\end{array}
$$

Virtual substitution: Example

$\exists x.\exists y.\ ((\ xy - 1 = 0 \quad \vee \quad y - x \geq 0\ ) \quad \wedge \quad y^2 - 1 < 0\ )$

Eliminate y: 2. Test candidate: $\frac{1}{x}$, if $x \neq 0$

Virtual substitution: Example

$\exists x.\exists y.\ ((\ xy - 1 = 0 \quad \vee \quad y - x \geq 0\ ) \quad \wedge \quad y^2 - 1 < 0\ )$

Eliminate y: 2. Test candidate: $\frac{1}{x}$, if $x \neq 0$

$$
\begin{array}{llll}
\exists x.(& (& & (xy-1=0)[\frac{1}{x}//y] \\
 & & \vee & (y-x \geq 0)[\frac{1}{x}//y] \quad) \\
 & \wedge & & (y^2-1<0)[\frac{1}{x}//y] \\
 & \wedge & & x \neq 0 \quad)
\end{array}
$$

Virtual substitution: Example

$\exists x.\exists y.\ ((\ xy - 1 = 0 \quad \vee \quad y - x \geq 0\ ) \quad \wedge \quad y^2 - 1 < 0\ )$

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$$
\begin{array}{llll}
\exists x.(& (& (xy - 1 = 0)[\frac{1}{x}//y] & \\
 & \vee & (y - x \geq 0)[\frac{1}{x}//y] &) \\
 \wedge & & (y^2 - 1 < 0)[\frac{1}{x}//y] & \\
 \wedge & & x \neq 0 &)
\end{array}
$$

$$
\begin{array}{llll}
\exists x.(& (& (0 = 0) & \\
 & \vee & ((x > 0 \ \wedge \ 1 - x^2 \geq 0) \ \vee \ (x < 0 \ \wedge \ 1 - x^2 \leq 0)) &) \\
 \wedge & & ((1 > 0 \ \wedge \ 1 - x^2 < 0) \ \vee \ (1 < 0 \ \wedge \ 1 - x^2 > 0)) & \\
 \wedge & & x \neq 0 &)
\end{array}
$$

Virtual substitution: Example

$\exists x.\exists y.\ ((\ xy-1=0 \ \vee \ y-x\geq 0\ ) \ \wedge \ y^2-1<0\ )$

Eliminate y: 2. Test candidate: $\frac{1}{x}$, if $x\neq 0$

$$
\begin{array}{lll}
\exists x.(& (& (xy-1=0)[\frac{1}{x}//y] \\
& \vee & (y-x\geq 0)[\frac{1}{x}//y] \quad) \\
& \wedge & (y^2-1<0)[\frac{1}{x}//y] \\
& \wedge & x\neq 0 \quad)
\end{array}
$$

$$
\begin{array}{lll}
\exists x.(& (& (0=0) \\
& \vee & ((x>0 \ \wedge\ 1-x^2\geq 0) \ \vee\ (x<0\ \wedge\ 1-x^2\leq 0)) \quad) \\
& \wedge & ((1>0\ \wedge\ 1-x^2<0) \ \vee\ (1<0\ \wedge\ 1-x^2>0)) \\
& \wedge & x\neq 0 \quad)
\end{array}
$$

$$
\begin{array}{ll}
\exists x.(& true \\
\wedge & 1-x^2<0 \\
\wedge & x\neq 0 \quad)
\end{array}
$$

Virtual substitution: Example

$\exists x.\ (((x > 0 \wedge 1 - x^2 \geq 0) \vee (x < 0 \wedge 1 - x^2 \leq 0)) \wedge 1 - x^2 < 0 \wedge x \neq 0)$

Eliminate x:

1. Test candidate: $-\infty$

$$\begin{aligned} & (x > 0)[-\infty//x] \\ = \quad & (1 < 0 \ \vee \ (1 = 0 \ \wedge \ 0 > 0)) \\ = \quad & \text{False} \end{aligned}$$

Virtual substitution: Example

$\exists x.\ (((\mathrm{False} \wedge 1 - x^2 \geq 0) \vee (x < 0 \wedge 1 - x^2 \leq 0)) \wedge 1 - x^2 < 0 \wedge x \neq 0)$

Eliminate x:

1. Test candidate: $-\infty$

$$
\begin{aligned}
& (x < 0)[-\infty//x] \\
= \quad & (1 > 0 \ \vee \ (1 = 0 \ \wedge \ 0 < 0)) \\
= \quad & \mathrm{True}
\end{aligned}
$$

Virtual substitution: Example

$\exists x.\ (((\text{False} \wedge 1 - x^2 \geq 0) \vee (\text{True} \wedge 1 - x^2 \leq 0)) \wedge 1 - x^2 < 0 \wedge x \neq 0)$

Eliminate x:

1. Test candidate: $-\infty$

$$\begin{aligned} & (1 - x^2 \leq 0)[-\infty//x] \\ = \ & (-1 < 0 \ \vee \ (-1 = 0 \ \wedge \ 1 > 0) \ \vee \ (-1 = 0 \ \wedge \ 1 = 0 \ \wedge \ 0 \leq 0)) \\ = \ & \text{True} \end{aligned}$$

Virtual substitution: Example

$\exists x.\ (((\text{False} \wedge 1 - x^2 \geq 0) \vee (\text{True} \wedge \text{True})) \wedge 1 - x^2 < 0 \wedge x \neq 0)$

Eliminate x:

1. Test candidate: $-\infty$

$$\begin{aligned} & (1 - x^2 < 0)[-\infty // x] \\ = \;& (-1 < 0 \;\vee\; (-1 = 0 \;\wedge\; 0 > 0) \;\vee\; (-1 = 0 \;\wedge\; 0 = 0 \;\wedge\; 1 < 0)) \\ = \;& \text{True} \end{aligned}$$

Virtual substitution: Example

$\exists x.\ (((\mathrm{False} \wedge 1 - x^2 \geq 0) \vee (\mathrm{True} \wedge \mathrm{True})) \wedge \mathrm{True} \wedge x \neq 0)$

Eliminate x:

1. Test candidate: $-\infty$

$$\begin{aligned} & (x \neq 0)[-\infty//x] \\ = \ & (1 \neq 0 \ \vee \ 0 \neq 0) \\ = \ & \mathrm{True} \end{aligned}$$

Virtual substitution: Example

$\exists x.\ (((\text{False} \wedge 1 - x^2 \geq 0) \vee (\text{True} \wedge \text{True})) \wedge \text{True} \wedge \text{True})$

Eliminate x:

1. Test candidate: $-\infty$

$$\begin{aligned} & (x \neq 0)[-\infty//x] \\ = \quad & (1 \neq 0 \ \vee \ 0 \neq 0) \\ = \quad & \text{True} \end{aligned}$$

Complexity

We consider in the following the elimination of one existential quantifier (existentially quantified variable):

$$\exists x_1. \ldots \exists x_n.\ \varphi' \quad \equiv \quad \exists x_1. \ldots \exists x_{n-1}. \bigvee_{t \in T} \varphi'[t//x_n].$$

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- Degree of a remaining variable x_i, $1 \leq i < n$, in φ', i.e. $D(x_i,\ \varphi')$:

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- Number of atoms in φ', i.e. $at(\varphi')$:

$$at(\bigvee_{t \in T} \varphi'[t//x_n]) \ \in\ \mathcal{O}(8at(\varphi') + at(\varphi')(8 + 63at(\varphi')))$$

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