

# Examples for the Omega Test

## 1 Example for the Omega test

Assume that we want to organize a game evening. We can arrange three types of games. The number of players per game are 4, 2 and 5 for the first, second respectively the third game type. We have to buy the games. A game of the first, second resp. third type costs 5, 4 respectively 6 units of money. We want to invite 50 guests and try to find an appropriate number of games of each type such that each guest can play in one of the games and we do not need to invest more than 60 units of money to buy the games.

The problem can be formalized as

$$4g_1 + 2g_2 + 5g_3 = 50 \quad (1)$$

$$5g_1 + 4g_2 + 6g_3 \leq 60 \quad (2)$$

$$g_1 \geq 0 \quad (3)$$

$$g_2 \geq 0 \quad (4)$$

$$g_3 \geq 0 \quad (5)$$

$$(6)$$

where  $g_1$ ,  $g_2$  resp.  $g_3$  denote the number of games of the first, second resp. third type.

### 1.1 Eliminating the equations

We have one equation without any coefficient 1 or  $-1$ . Therefore we use our coefficient reduction technique to generate a new equation. The smallest coefficient is 2 thus we apply  $\cdot \widehat{\text{mod}} 3$ :

$$(4 \widehat{\text{mod}} 3)g_1 + (2 \widehat{\text{mod}} 3)g_2 + (5 \widehat{\text{mod}} 3)g_3 = 3\sigma + (50 \widehat{\text{mod}} 3)$$

$$(4 - 3\lfloor 4/3 + 1/2 \rfloor)g_1$$

$$+(2 - 3\lfloor 2/3 + 1/2 \rfloor)g_2$$

$$+(5 - 3\lfloor 5/3 + 1/2 \rfloor)g_3 = 3\sigma + (50 - 3\lfloor 50/3 + 1/2 \rfloor)$$

$$1 \cdot g_1 + (-1) \cdot g_2 + (-1) \cdot g_3 = 3\sigma + (-1)$$

$$g_2 = g_1 - g_3 - 3\sigma + 1$$

We add this equation to our problem and use it to eliminate  $g_2$ , resulting in the problem

$$\begin{aligned} g_1 &\geq 0 \\ g_1 - g_3 - 3\sigma &\geq -1 \\ g_3 &\geq 0 \\ g_3 &= -2g_1 + 2\sigma + 16 \\ 9g_1 + 2g_3 - 12\sigma &\leq 56 \end{aligned}$$

Next we eliminate  $g_3$  using the only equation. This gives us

$$\begin{aligned} g_1 &\geq 0 \\ 3g_1 - 5\sigma &\geq 15 \\ -g_1 + \sigma &\geq -8 \\ 5g_1 - 8\sigma &\leq 24 \end{aligned}$$

Now we have only inequations and we can start with the real shadow of the Omega test.

## 1.2 Real shadow

We first eliminate  $g_1$ . We have two lower bounds

$$\begin{aligned} 0 &\leq g_1 \\ 5\sigma + 15 &\leq 3g_1 \end{aligned}$$

and two upper bounds

$$\begin{aligned} g_1 &\leq \sigma + 8 \\ 5g_1 &\leq 8\sigma + 24 \end{aligned}$$

For the first lower and first upper bound we get

$$\begin{aligned} 0 &\leq \sigma + 8 \\ \Leftrightarrow -8 &\leq \sigma \end{aligned}$$

For the second lower and the first upper bound

$$\begin{aligned} 5\sigma + 15 &\leq 3\sigma + 24 \\ \Leftrightarrow 2\sigma &\leq 9 \\ \Leftrightarrow \sigma &\leq 4 \end{aligned}$$

For the first lower and the second upper bound

$$\begin{aligned} 0 &\leq 8\sigma + 24 \\ \Leftrightarrow -24 &\leq 8\sigma \\ \Leftrightarrow -3 &\leq \sigma \end{aligned}$$

Finally, for the second lower and second upper bound

$$\begin{aligned} 25\sigma + 75 &\leq 24\sigma + 72 \\ \Leftrightarrow \sigma &\leq -3 \end{aligned}$$

These constraints define the interval  $[-3, -3]$  for  $\sigma$  which contains the integer  $-3$ , thus the real shadow returns without any conflict.

### 1.3 Dark shadow

*Reminder:* We can eliminate a variable  $x$  by introducing for each lower-upper-bound pair

$$\beta \leq bx, \quad cx \leq \gamma$$

a new constraint

$$b\gamma - c\beta \leq (c-1)(b-1).$$

Afterwards we can discard all constraints containing  $x$ .

We start eliminating the variable  $g_1$ , which is constrained by the two lower bounds

$$\begin{aligned} 0 &\leq g_1 \\ 5\sigma + 15 &\leq 3g_1 \end{aligned}$$

and two upper bounds

$$\begin{aligned} g_1 &\leq \sigma + 8 \\ 5g_1 &\leq 8\sigma + 24. \end{aligned}$$

For the first lower bound and the first upper bound we get

$$\begin{aligned} 1 \cdot (\sigma + 8) - 1 \cdot 0 &\geq (1-1)(1-1) \\ \Leftrightarrow \sigma + 8 &\geq 0 \\ \Leftrightarrow \sigma &\geq -8. \end{aligned}$$

For the second lower bound and the second upper bound we get

$$\begin{aligned}
& 3 \cdot (8\sigma + 24) - 5 \cdot (5\sigma + 15) \geq (5 - 1)(3 - 1) \\
\Leftrightarrow & \quad 24\sigma + 72 - 25\sigma - 75 \geq 8 \\
\Leftrightarrow & \quad \quad \quad -\sigma \geq 11 \\
\Leftrightarrow & \quad \quad \quad \sigma \leq -11.
\end{aligned}$$

We get the conflict  $-8 \leq \sigma \leq -11$  and know that the dark shadow contains no integer solution for  $g_1$ . Hence, we continue with the gray shadow.

## 1.4 Gray shadow

*Reminder:* For a lower-upper-bound pair  $\beta \leq bx$  and  $cx \leq \gamma$  try all possible values  $i$  with  $bx = \beta + i$ ,  $0 \leq i \leq \frac{cb-c-b}{c}$ . For each of these values we extend the constraint set with the equation  $bx = \beta + i$  and check it for satisfiability. We first eliminate  $g_1$ . The lower bounds on  $g_1$  are

$$\begin{aligned}
0 & \leq g_1 \\
5\sigma + 15 & \leq 3g_1
\end{aligned}$$

and the upper bounds on  $g_1$  are

$$\begin{aligned}
g_1 & \leq \sigma + 8 \\
5g_1 & \leq 8\sigma + 24.
\end{aligned}$$

For the first lower bound and the first upper bound we get

$$g_1 = i, \quad 0 \leq i \leq -\frac{1}{5}$$

resulting in an empty set of cases.

For the first lower bound and the second upper bound we get

$$3g_1 = 5\sigma + 15 + i, \quad 0 \leq i \leq \left\lfloor \frac{5 \cdot 3 - 5 - 3}{5} \right\rfloor = 1$$

resulting in the following cases:

- i)  $3g_1 = 5\sigma + 15$
- ii)  $3g_1 = 5\sigma + 16$

Considering the first case we extend the system to

$$\begin{aligned}
g_1 &\geq 0 \\
3g_1 - 5\sigma &\geq 15 \\
-g_1 + \sigma &\geq -8 \\
5g_1 - 8\sigma &\leq 24 \\
3g_1 - 5\sigma &= 15
\end{aligned}$$

We use the equation  $3g_1 = 5\sigma + 15$  to eliminate  $g_1$ . The smallest coefficient occurring is 3, therefore we apply  $(\cdot \widehat{\text{mod}} 4)$ :

$$\begin{aligned}
(3 \widehat{\text{mod}} 4)g_1 - (5 \widehat{\text{mod}} 4)\sigma &= (15 \widehat{\text{mod}} 4) + 4\sigma_2 \\
\Leftrightarrow (-1)g_1 - (1)\sigma &= (-1) + 4\sigma_2 \\
\Leftrightarrow g_1 &= -\sigma - 4\sigma_2 + 1
\end{aligned}$$

and substitute  $g_1$  by  $-\sigma - 4\sigma_2 + 1$  in all constraints.

1. In  $g_1 \geq 0$ :

$$\begin{aligned}
-\sigma - 4\sigma_2 + 1 &\geq 0 \\
\Leftrightarrow \sigma + 4\sigma_2 &\leq 1
\end{aligned}$$

2. In  $3g_1 - 5\sigma \geq 15$ :

$$\begin{aligned}
3(-\sigma - 4\sigma_2 + 1) - 5\sigma &\geq 15 \\
-8\sigma - 12\sigma_2 &\geq 12 \\
\Leftrightarrow 2\sigma + 3\sigma_2 &\leq -3
\end{aligned}$$

3. In  $-g_1 + \sigma \geq -8$ :

$$\begin{aligned}
-(-\sigma - 4\sigma_2 + 1) + \sigma &\geq -8 \\
2\sigma + 4\sigma_2 &\geq -7
\end{aligned}$$

4. In  $5g_1 - 8\sigma \leq 24$ :

$$\begin{aligned}
5(-\sigma - 4\sigma_2 + 1) - 8\sigma &\leq 24 \\
-13\sigma - 20\sigma_2 &\leq 19 \\
13\sigma + 20\sigma_2 &\geq -19
\end{aligned}$$

5. In  $3g_1 - 5\sigma = 15$ :

$$\begin{aligned} 3(-\sigma - 4\sigma_2 + 1) &= 5\sigma + 15 \\ -8\sigma - 12\sigma_2 &= 12 \\ 2\sigma + 3\sigma_2 &= -3 \end{aligned}$$

Thus, we still have a equation, but with a smaller greatest coefficient than the equation we have just used to eliminate  $g_1$ . We continue with the new equation  $2\sigma + 3\sigma_2 = -3$  to eliminate the variable with the smallest coefficient, i.e.  $\sigma$ . Note, that we eventually get rid of the equation. The smallest coefficient occurring now is 2, therefore we apply  $(\cdot \widehat{\text{mod}} 3)$ :

$$\begin{aligned} (-2 \widehat{\text{mod}} 3)\sigma - (3 \widehat{\text{mod}} 3)\sigma &= (3 \widehat{\text{mod}} 3) + 3\sigma_3 \\ \Leftrightarrow (1)g_1 - (0)\sigma &= (0) + 3\sigma_3 \\ \Leftrightarrow &\sigma = 3\sigma_3 \end{aligned}$$

and substitute  $\sigma$  by  $3\sigma_3$  in all constraints.

1. In  $\sigma + 4\sigma_2 \leq 1$ :

$$\begin{aligned} -3\sigma_3 - 4\sigma_2 + 1 &\geq 0 \\ \Leftrightarrow 3\sigma_3 + 4\sigma_2 &\leq 1 \end{aligned}$$

2. In  $2\sigma + 3\sigma_2 \leq -3$ :

$$\begin{aligned} 6\sigma_3 + 3\sigma_2 &\leq -3 \\ \Leftrightarrow 2\sigma_3 + \sigma_2 &\leq -1 \end{aligned}$$

3. In  $2\sigma + 4\sigma_2 \geq -7$ :

$$6\sigma_3 + 4\sigma_2 \geq -7$$

4. In  $13\sigma + 20\sigma_2 \geq -19$ :

$$39\sigma_3 + 20\sigma_2 \geq -19$$

5. In  $2\sigma + 3\sigma_2 = -3$ :

$$\begin{aligned} 6\sigma_3 + 3\sigma_2 &= -3 \\ \Leftrightarrow \sigma_2 &= -2\sigma_3 - 1 \end{aligned}$$

Now, we eliminate  $\sigma_2$  using  $\sigma_2 = -2\sigma_3 - 1$  and get

1. In  $3\sigma_3 + 4\sigma_2 \leq 1$ :

$$\begin{aligned} 3\sigma_3 - 8\sigma_2 &\leq 5 \\ \Leftrightarrow \sigma_3 &\geq -1 \end{aligned}$$

2. In  $2\sigma_3 + \sigma_2 \leq -1$ :

$$\begin{aligned} -2\sigma_3 + 2\sigma_3 + 1 &\geq 1 \\ \Leftrightarrow 1 &\geq 1 \end{aligned}$$

3. In  $6\sigma_3 + 4\sigma_2 \geq -7$ :

$$\begin{aligned} 6\sigma_3 + 4(-2\sigma_3 - 1) &\geq -7 \\ \Leftrightarrow -2\sigma_3 &\geq -3 \\ \Leftrightarrow \sigma_3 &\leq 1 \end{aligned}$$

4. In  $39\sigma_3 + 20\sigma_2 \geq -19$ :

$$\begin{aligned} 39\sigma_3 + 20(-2\sigma_3 - 1) &\geq -19 \\ \Leftrightarrow -\sigma_3 &\geq 1 \\ \Leftrightarrow \sigma_3 &\leq -1 \end{aligned}$$

5. In  $2\sigma_3 + \sigma_2 = -1$ :

$$\begin{aligned} 2\sigma_3 - 2\sigma_3 - 1 &= -1 \\ \Leftrightarrow -1 &= -1 \end{aligned}$$

So  $\sigma_3 = -1$  is a solution. Using the equations we can find a satisfying assignment as follows. From  $\sigma_2 = -2\sigma_3 - 1$  we get  $\sigma_2 = 1$ , from  $\sigma = 3\sigma_3$  we get  $\sigma = -3$ , from  $g_1 = -\sigma - 4\sigma_2 + 1$  we get

$$g_1 = 0,$$

from  $g_3 = -2g_1 + 2\sigma + 16$  we get

$$g_3 = 10,$$

and from  $g_2 = g_1 - g_3 - 3\sigma + 1$  we get

$$g_2 = 0.$$

Considering the original problem

$$4g_1 + 2g_2 + 5g_3 = 50$$

$$5g_1 + 4g_2 + 6g_3 \leq 60$$

$$g_1 \geq 0$$

$$g_2 \geq 0$$

$$g_3 \geq 0$$

you can see that it is indeed a satisfying assignment.