Satisfiability Checking

The Omega Test

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Informatik 2
LuFG Theory of Hybrid Systems

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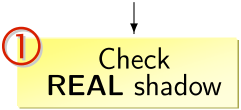
The Omega test

- Goal: Decide satisfiability for conjunctions of linear constraints of the form

$$\sum_{0 \leq i \leq n} a_i x_i \geq b$$

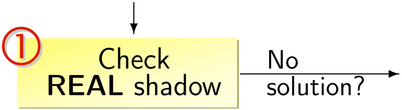
  over integers.
- Original application:
  Program optimizations done by a compiler.
- Extension of *Fourier-Motzkin* variable elimination:
  - Pick one variable and eliminate it.
  - Continue until all variables but one are eliminated.

Checks for real solution with an integer value in a given dimension.



Overview of the Omega test

Checks for real solution
with an integer value
in a given dimension.



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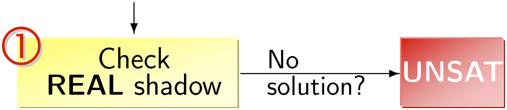
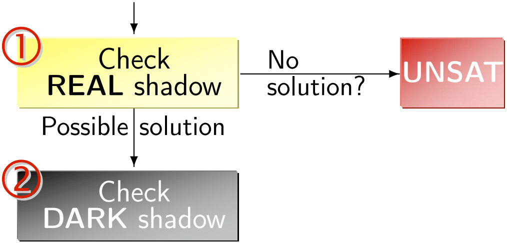


Overview of the Omega test

Checks for real solution with an integer value in a given dimension.

Checks a sufficient condition for integer solution.



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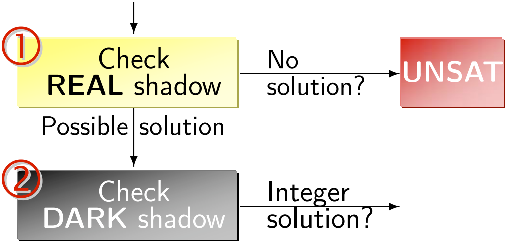

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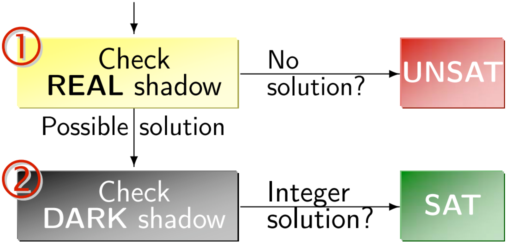
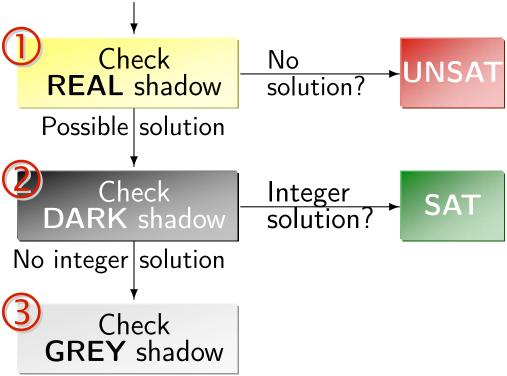


Overview of the Omega test

Checks for real solution with an integer value in a given dimension.

Checks a sufficient condition for integer solution.

Checks for integer solutions not satisfying the sufficient condition.



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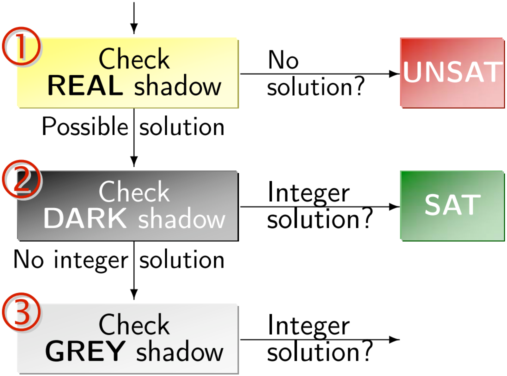

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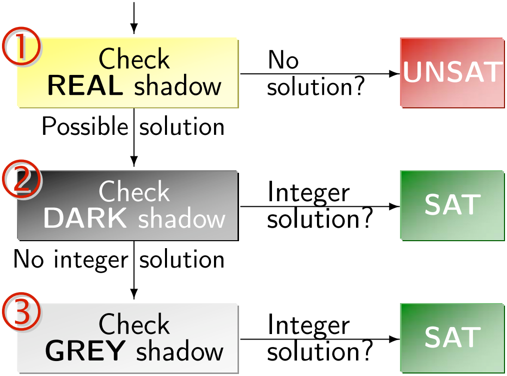

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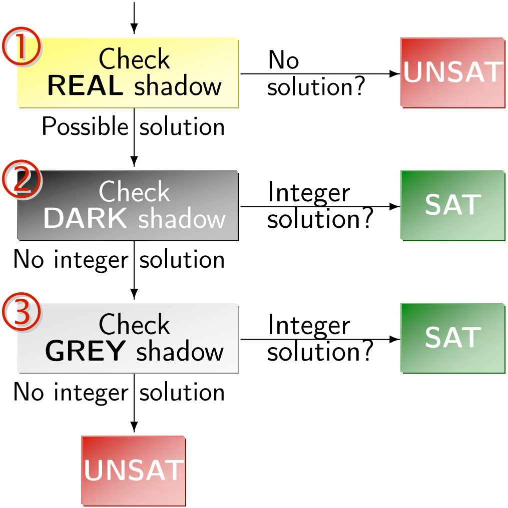

The real shadow

① Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\beta \quad \leq \quad bz \qquad cz \quad \leq \quad \gamma \quad (b, c > 0)$$

The real shadow

① Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\begin{array}{rclcrclc} \beta & \leq & bz & \quad & cz & \leq & \gamma & \quad (b, c > 0) \\ c\beta & \leq & cbz & & cbz & \leq & b\gamma & \end{array}$$

Important: All terms are integer-valued. (Instead of cb, we can also use the smallest common multiple of c and b.)

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- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\begin{array}{rclcrclc} \beta & \leq & bz & \quad & cz & \leq & \gamma & \quad (b, c > 0) \\ c\beta & \leq & cbz & \quad & cbz & \leq & b\gamma & \end{array}$$

Important: All terms are integer-valued. (Instead of cb, we can also use the smallest common multiple of c and b.)

- Constraint for real shadow if z is not the last variable to be eliminated:

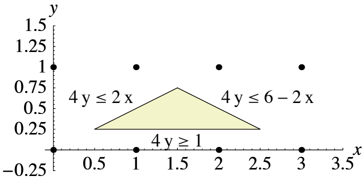
$$c\beta \leq b\gamma$$

- Constraint for real shadow if z is the last variable to be eliminated:

$$\left\lceil \frac{\beta}{b} \right\rceil \leq \left\lfloor \frac{\gamma}{c} \right\rfloor$$

The real shadow: Example I

① Check **REAL** shadow



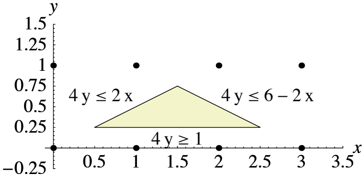
$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

The real shadow: Example I

① Check **REAL** shadow



$4y \leq 2x$

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Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

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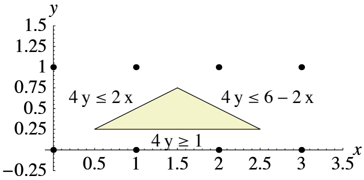

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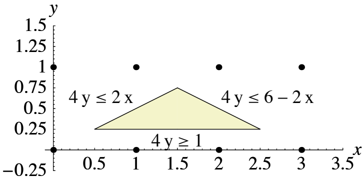
Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$$4y \leq 6 - 4y$$

The real shadow: Example I

① Check **REAL** shadow



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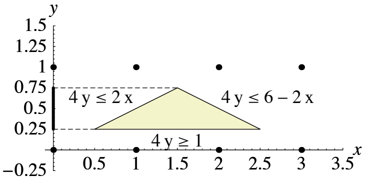
Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$$4y \leq 6 - 4y$$
$$8y \leq 6$$

The real shadow: Example I

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$4y \leq 6 - 4y$

$8y \leq 6$

Real Shadow:

$8y \leq 6$

$4y \geq 1$

$\Rightarrow$

$y \leq 0.75$

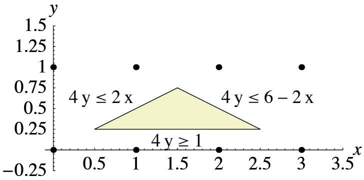
$y \geq 0.25$

No integer solution

$\Longrightarrow$ Original problem has no solution

The real shadow: Example II

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

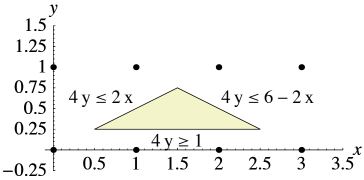
$4y \geq 1$

Let's eliminate y instead:

$1 \leq 4y$ $\quad$ $4y \leq 2x$

The real shadow: Example II

1 Check **REAL** shadow



$4y \leq 2x$

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Let's eliminate y instead:

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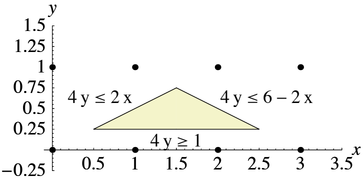
$1 \leq 2x$

1 Check **REAL** shadow



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Let's eliminate y instead:

$1 \leq 4y$ $\quad$ $4y \leq 2x$

$1 \leq 2x$

$1 \leq 4y$ $\quad$ $4y \leq -2x + 6$

The real shadow: Example II

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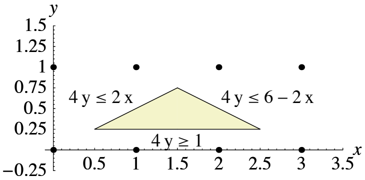


$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

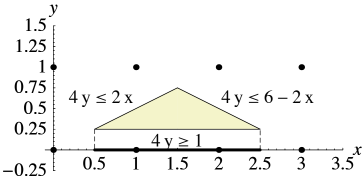
Let's eliminate y instead:

$1 \leq 4y$ and $4y \leq 2x$ → $1 \leq 2x$

$1 \leq 4y$ and $4y \leq -2x + 6$ → $1 \leq -2x + 6$

The real shadow: Example II

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

Let's eliminate y instead:

$1 \leq 4y$ and $4y \leq 2x$ $\Rightarrow$ $1 \leq 2x$

$1 \leq 4y$ and $4y \leq -2x + 6$ $\Rightarrow$ $1 \leq -2x + 6$

Real Shadow:

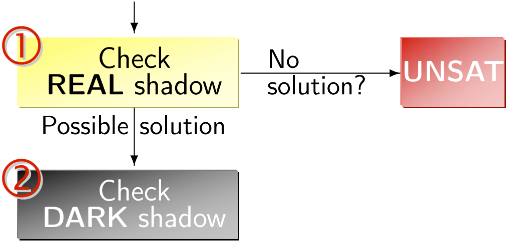
$1 \leq 2x$ $\Rightarrow$ $x \geq 0.5$

$1 \leq -2x + 6$ $\Rightarrow$ $x \leq 2.5$

Real solution with integer x-value!

But there exists
no pure integer solution!

From real to dark shadow



- A solution for the REAL shadow does not guarantee that there is an integer solution for the original problem.
- Thus, we check the DARK shadow next.

Idea of the dark shadow

② Check **DARK** shadow

- Idea of the DARK shadow:

$$\beta \leq bz \qquad\qquad cz \leq \gamma$$

Idea of the dark shadow

② Check **DARK** shadow

- Idea of the DARK shadow:

$$\begin{array}{rclcrclcl} \beta & \leq & bz & \mid : b & cz & \leq & \gamma & \mid : c & \\ \frac{\beta}{b} & \leq & z & & z & \leq & \frac{\gamma}{c} & & z \in \mathbb{N} \end{array}$$

Idea of the dark shadow

② Check **DARK** shadow

- Idea of the DARK shadow:

$$\begin{array}{rclcrclc} \beta & \leq & bz & \mid : b & cz & \leq & \gamma & \mid : c \\ \frac{\beta}{b} & \leq & z & & z & \leq & \frac{\gamma}{c} & \qquad z \in \mathbb{N} \end{array}$$

- Try to prove that there is an integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$.

Dark shadow: Proof by contradiction

② Check
DARK shadow

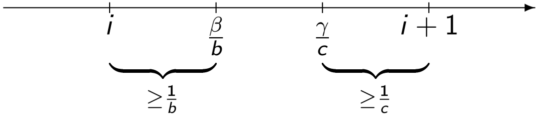
Assume there is no integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$.

Dark shadow: Proof by contradiction

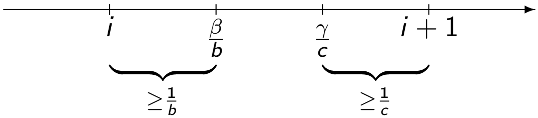
② Check **DARK** shadow

Assume there is no integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$. Then:

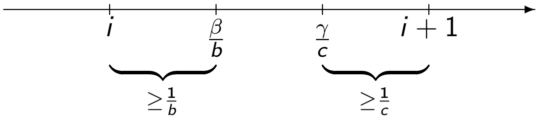
Let $i := \lfloor \frac{\beta}{b} \rfloor$ $\quad i \in \mathbb{Z}$



Dark shadow: Proof by contradiction

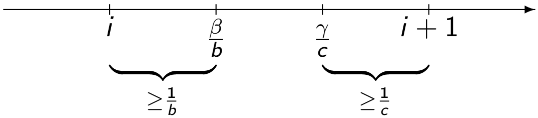


Dark shadow: Proof by contradiction



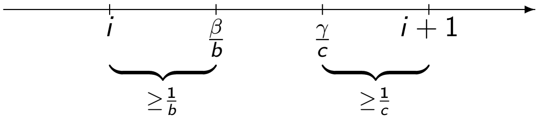
$$\begin{aligned} \frac{\beta}{b} - i &\geq \frac{1}{b} \\ i + 1 - \frac{\gamma}{c} &\geq \frac{1}{c} \end{aligned}$$

Dark shadow: Proof by contradiction

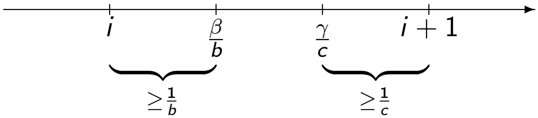


$$\begin{array}{rcl} \frac{\beta}{b} - i & \geq & \frac{1}{b} \\ i + 1 - \frac{\gamma}{c} & \geq & \frac{1}{c} \\ \hline \hline \frac{\beta}{b} + 1 - \frac{\gamma}{c} & \geq & \frac{1}{b} + \frac{1}{c} \end{array}$$

Dark shadow: Proof by contradiction



$$
\begin{array}{rcll}
\frac{\beta}{b} - i & \geq & \frac{1}{b} & \\
i + 1 - \frac{\gamma}{c} & \geq & \frac{1}{c} & \\
\hline\hline
\frac{\beta}{b} + 1 - \frac{\gamma}{c} & \geq & \frac{1}{b} + \frac{1}{c} & \quad | \cdot c \cdot b \\
c\beta + cb - b\gamma & \geq & c + b &
\end{array}
$$



$$\frac{\beta}{b} - i \geq \frac{1}{b}$$

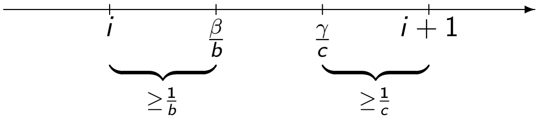
$$i + 1 - \frac{\gamma}{c} \geq \frac{1}{c}$$

$$\frac{\beta}{b} + 1 - \frac{\gamma}{c} \geq \frac{1}{b} + \frac{1}{c} \qquad | \cdot c \cdot b$$

$$c\beta + cb - b\gamma \geq c + b \qquad | - cb$$

$$c\beta - b\gamma \geq -cb + c + b$$

Dark shadow: Proof by contradiction



$$\begin{array}{rcll}
\frac{\beta}{b} - i & \geq & \frac{1}{b} & \\
i + 1 - \frac{\gamma}{c} & \geq & \frac{1}{c} & \\
\hline\hline
\frac{\beta}{b} + 1 - \frac{\gamma}{c} & \geq & \frac{1}{b} + \frac{1}{c} & | \cdot c \cdot b \\
c\beta + cb - b\gamma & \geq & c + b & | - cb \\
c\beta - b\gamma & \geq & -cb + c + b & | \cdot (-1)
\end{array}$$

$$b\gamma - c\beta \leq cb - c - b$$

Dark shadow: Proof by contradiction

- From previous slide:

$$b\gamma - c\beta \quad \leq \quad cb - c - b$$

Dark shadow: Proof by contradiction

- From previous slide:

$$\begin{array}{rrcl} & b\gamma - c\beta & \leq & cb - c - b \\ \leftrightarrow & \neg(b\gamma - c\beta & > & cb - c - b) \end{array}$$

Dark shadow: Proof by contradiction

- From previous slide:

$$
\begin{array}{rrcl}
 & b\gamma - c\beta & \leq & cb - c - b \\
\leftrightarrow & \neg(b\gamma - c\beta & > & cb - c - b) \\
\leftrightarrow & \neg(b\gamma - c\beta & \geq & cb - c - b + 1)
\end{array}
$$

Dark shadow: Proof by contradiction

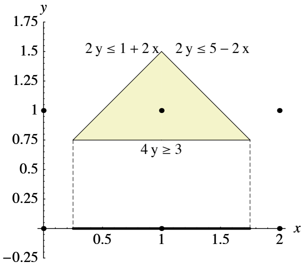
- From previous slide:

$$
\begin{array}{lrcl}
 & b\gamma - c\beta & \leq & cb - c - b \\
\leftrightarrow & \neg(b\gamma - c\beta & > & cb - c - b) \\
\leftrightarrow & \neg(b\gamma - c\beta & \geq & cb - c - b + 1) \\
\leftrightarrow & \neg(\underbrace{b\gamma - c\beta \geq (c-1)(b-1)}_{*}) & &
\end{array}
$$

- Thus, if * holds, we know that there must be an integer solution.
- If $c = 1$ or $b = 1$, then this is the same as the real shadow.
  This case is called an exact projection.

Example for the dark shadow

② Check DARK shadow



$2y \leq 2x + 1$

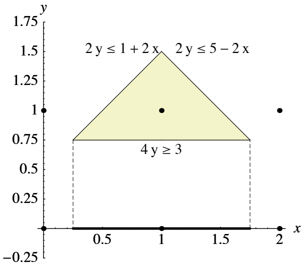
$2y \leq -2x + 5$

$4y \geq 3$

Example for the dark shadow

Reminder: $(\beta \leq bz \land cz \leq \gamma) \rightarrow b\gamma - c\beta \geq (c-1)(b-1)$

② Check DARK shadow



$2y \leq 2x + 1$

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$4y \geq 3$

;

Eliminate y with the dark shadow:

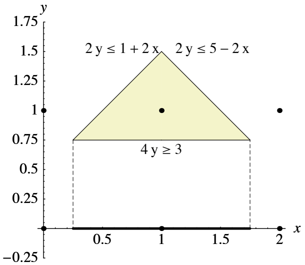
$2y \leq 2x + 1$ $\quad$ $4y \geq 3$

$$4(2x+1) - 2 \cdot 3 \geq (2-1)(4-1)$$

Example for the dark shadow

Reminder: $(\beta \leq bz \wedge cz \leq \gamma) \rightarrow b\gamma - c\beta \geq (c-1)(b-1)$

② Check DARK shadow



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$2y \leq 2x + 1$ $\quad$ $4y \geq 3$

$$4(2x+1) - 2 \cdot 3 \geq (2-1)(4-1)$$

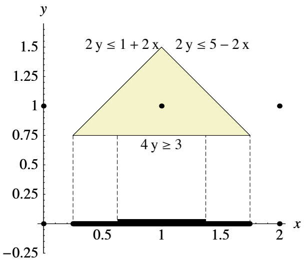
$4y \geq 3$ $\quad$ $2y \leq -2x + 5$

$$4(-2x+5) - 2 \cdot 3 \geq (2-1)(4-1)$$

Example for the dark shadow

Reminder: $(\beta \leq bz \wedge cz \leq \gamma) \rightarrow b\gamma - c\beta \geq (c-1)(b-1)$

② Check DARK shadow



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Eliminate y with the dark shadow:

$2y \leq 2x+1$ $\quad$ $4y \geq 3$

$$4(2x+1) - 2 \cdot 3 \geq (2-1)(4-1)$$

$4y \geq 3$ $\quad$ $2y \leq -2x+5$

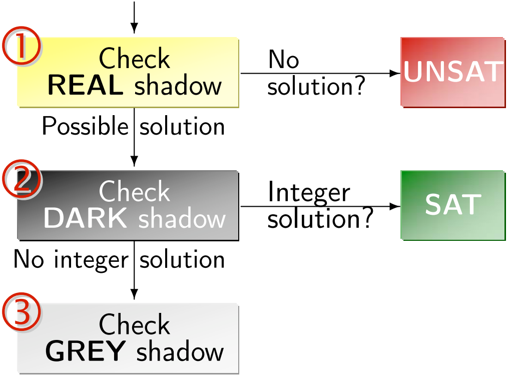
$$4(-2x+5) - 2 \cdot 3 \geq (2-1)(4-1)$$

Dark Shadow:

$$x \geq 5/8$$
$$x \leq 11/8$$

$\Longrightarrow$ Integer solution!

From dark to grey shadow



- No integer solution in the DARK shadow does not guarantee that there is no integer solution for the original problem.
- Thus, we check the GREY shadow next.

The grey shadow

③ Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

The grey shadow

③ Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

$$
\begin{array}{rl}
\text{In R:} & b\gamma \geq cbz \geq c\beta \\
\textbf{Not} \text{ in D:} & cb - c - b \geq b\gamma - c\beta \\
\leftrightarrow & cb - c - b + c\beta \geq b\gamma \\
\Rightarrow & cb - c - b + c\beta \geq cbz \geq c\beta
\end{array}
$$

The grey shadow

③ Check
GREY shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

$$\begin{array}{rll}
\text{In R:} & b\gamma \geq cbz \geq c\beta & \\
\textbf{Not} \text{ in D:} & cb - c - b \geq b\gamma - c\beta & \\
\leftrightarrow & cb - c - b + c\beta \geq b\gamma & \\
\Rightarrow & cb - c - b + c\beta \geq cbz \geq c\beta & \mid : c \\
 & (cb - c - b)/c + \beta \geq bz \geq \beta &
\end{array}$$

The grey shadow

③ Check **GREY** shadow

- Try all values of z such that

$$(cb - c - b)/c + \beta \geq bz \geq \beta$$

The grey shadow

③ Check **GREY** shadow

- Try all values of z such that

$$(cb - c - b)/c + \beta \geq bz \geq \beta$$

- Optimization: find the largest coefficient c in any upper bound and try the following for each lower bound $bz \geq \beta$:

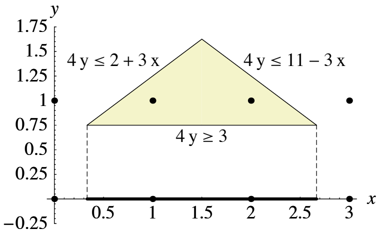
$$bz = \beta + i \quad \text{for } 0 \leq i \leq (cb - c - b)/c$$

- As before, combine this with the original problem, and solve recursively.

Example of the grey shadow

Reminder: $bz = \beta + i \quad$ for $0 \leq i \leq (cb - c - b)/c$

3 Check **GREY** shadow



$$4y \leq 3x + 2$$

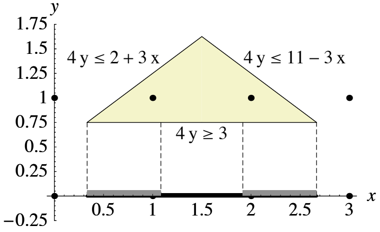
$$4y \leq -3x + 11$$

$$4y \geq 3$$

Example of the grey shadow

Reminder: $bz = \beta + i \quad$ for $0 \le i \le (cb - c - b)/c$

③ Check **GREY** shadow



$$4y \le 3x + 2$$

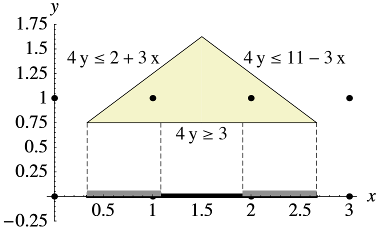
$$4y \le -3x + 11$$

$$4y \ge 3$$

Example of the grey shadow

Reminder: $bz = \beta + i \quad$ for $0 \leq i \leq (cb - c - b)/c$

③ Check **GREY** shadow



$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

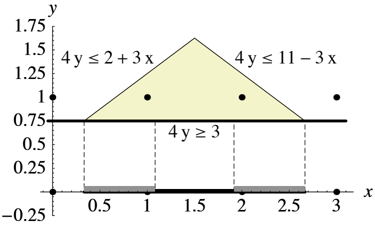
$$4y \geq 3$$

- Eliminate y:
  $c = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i \quad$ for
  $2 \geq i \geq 0$:

Example of the grey shadow

Reminder: $bz = \beta + i \quad$ for $0 \le i \le (cb - c - b)/c$

③ Check **GREY** shadow



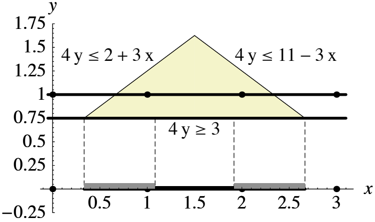
$$4y \le 3x + 2$$
$$4y \le -3x + 11$$
$$4y \ge 3$$

- Eliminate y:
  $c = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i$ for
  $2 \ge i \ge 0$:
  $$4y = 3$$

Example of the grey shadow

Reminder: $bz = \beta + i \quad$ for $0 \leq i \leq (cb - c - b)/c$

③ Check **GREY** shadow



$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

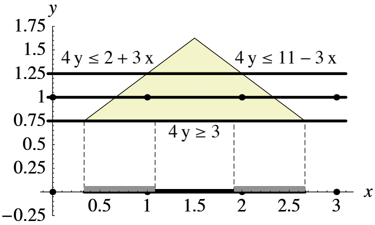
$$4y \geq 3$$

- Eliminate y:
  $c = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i \quad$ for
  $2 \geq i \geq 0$:
  $$4y = 3$$
  $$4y = 4$$

Example of the grey shadow

Reminder: $bz = \beta + i \quad$ for $0 \leq i \leq (cb - c - b)/c$

③ Check **GREY** shadow



$$4y \leq 3x + 2$$
$$4y \leq -3x + 11$$
$$4y \geq 3$$

- Eliminate y:
  $c = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i \quad$ for
  $2 \geq i \geq 0$:
  $4y = 3$
  $4y = 4$
  $4y = 5$

$\Longrightarrow$ Integer solution with $4y = 4$