Satisfiability Checking

Gomory Cuts

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Informatik 2
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WS 19/20

Branch and bound

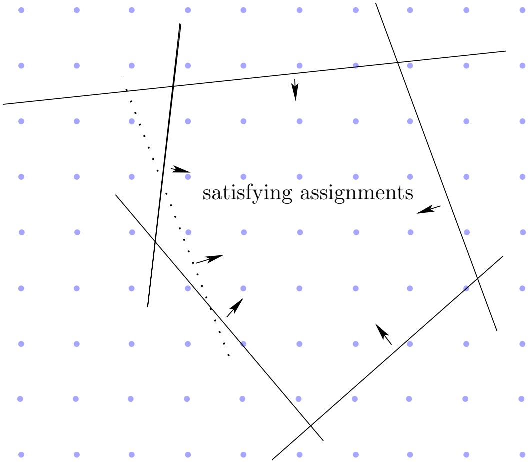
We use Simplex to find a real solution. If the solution is not integer-valued, we generate a new constraint such that the new (reduced) feasible region has two important properties:

- It does not contain the found non-integer solution any more.
- It still contains all feasible solutions to the original ILP problem.

Branch and bound

- We looked at branching by dividing the value domain of an integer variable into two halfs (branching).
- We could also cut with other, better contraints.
- E.g., for $x \in \mathbb{Z}$, from $2x \leq 11$ we can conclude $x \leq 5$.
- But how to generate such cutting planes?
- We look at one method for generating cutting planes: Gomory cuts.

Cutting planes, geometrically



The dotted line is a cutting plane.

Example: Gomory cuts

Suppose our input integer linear problem has

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After solving the relaxed problem:

- The final tableau of the general Simplex algorithm includes the constraint

$$x_3 = 0.5x_1 + 2.5x_2$$

- and the solution α is

$$\{x_3 \mapsto 1.75, x_1 \mapsto 1, x_2 \mapsto 0.5\}$$

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- Subtracting these values from the variables gives us

$$x_3 - 1.75 = 0.5(x_1 - 1) + 2.5(x_2 - 0.5) \ .$$

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- This constraint is unsatisfied by α because $\alpha(x_1) = 1$ and $\alpha(x_2) = 0.5$.
- Hence, this constraint removes the current solution.
- On the other hand, it is implied by the integer system of constraints, and hence cannot remove any *integer* solution.

Gomory cuts

- Generalizing this example:
  - Upper bounds.
  - Both positive and negative coefficients.
- The description that follows is based on
  - *Integrating Simplex with DPLL(T)*
    Technical report SRI-CSL-06-01
    Dutertre and de Moura (2006).

Gomory cuts

There are two preliminary conditions for deriving a Gomory cut from a constraint:

- The assignment to at least one basic or original variable is fractional.
- The non-basic variables are either additional variables or their coefficients are integers.
- One more constraint which we discuss later.

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- Consider the i-th constraint

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  with $x_i \in \mathcal{B}$, $\alpha(x_i)$ not an integer and a_{ij} integer for all $j \in \mathcal{N}_o$.

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$$\underbrace{x_i - \sum_{x_j \in \mathcal{N}_o} a_{ij} x_j}_{T} = \sum_{x_j \in \mathcal{N}_a} a_{ij} x_j$$

  Note: T is integer-valued.

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  Assumption: $\alpha(T)$ is not an integer.

Gomory cuts

- We have

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- Then also

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- It follows that

$$f_T + \sum_{j \in \mathcal{N}_a} a_{ij}(x_j - \alpha(x_j))$$

should be integer-valued. Note: $0 < f_T < 1$.

Gomory cuts

- Partition the non-basic additional variables to
  - those that are currently assigned their lower bound, and
  - those that are currently assigned their upper bound:

$$
\begin{aligned}
L &= \{j \mid x_j \in \mathcal{N}_a \wedge \alpha(x_j) = l_j\} \\
U &= \{j \mid x_j \in \mathcal{N}_a \wedge \alpha(x_j) = u_j\}\ .
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- We further split L and U as follows:

$$
\begin{array}{lcl}
L^+ & = & \{j \mid j \in L \wedge a_{ij} > 0\} \\
L^- & = & \{j \mid j \in L \wedge a_{ij} < 0\} \\
U^+ & = & \{j \mid j \in U \wedge a_{ij} > 0\} \\
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- and further equals

$$\begin{aligned} f_T \quad + & \quad \sum_{j \in L^+} a_{ij}(x_j - l_j) + \sum_{j \in L^-} a_{ij}(x_j - l_j) \\ - & \quad \sum_{j \in U^-} a_{ij}(u_j - x_j) - \sum_{j \in U^+} a_{ij}(u_j - x_j) \end{aligned}$$

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positive and integer-valued, thus

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- Gathering the positive components,
$$\sum_{j\in L^+} a_{ij}(x_j - l_j) - \sum_{j\in U^-} a_{ij}(u_j - x_j) \geq 1 - f_T \,,$$
or, equivalently,
$$\sum_{j\in L^+} \frac{a_{ij}}{1 - f_T}(x_j - l_j) - \sum_{j\in U^-} \frac{a_{ij}}{1 - f_T}(u_j - x_j) \geq 1 \,.$$

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Case 2: $\sum_{j \in L} a_{ij}(x_j - l_j) - \sum_{j \in U} a_{ij}(u_j - x_j) \leq 0$

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- Then
$$f_T + \sum_{j \in L} a_{ij}(x_j - l_j) - \sum_{j \in U} a_{ij}(u_j - x_j) \leq 0 \; .$$
- Gathering the negative components,
$$\sum_{j \in L^-} a_{ij}(x_j - l_j) - \sum_{j \in U^+} a_{ij}(u_j - x_j) \leq -f_T \; .$$

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- Dividing by $-f_T$ gives us
$$-\sum_{j\in L^-} \frac{a_{ij}}{f_T}(x_j - l_j) + \sum_{j\in U^+} \frac{a_{ij}}{f_T}(u_j - x_j) \geq 1\ .$$

Gomory cuts

- Case 1: $\sum_{j\in L} a_{ij}(x_j - l_j) - \sum_{j\in U} a_{ij}(u_j - x_j) > 0$:

$$\sum_{j\in L^+} \frac{a_{ij}}{1-f_T}(x_j - l_j) - \sum_{j\in U^-} \frac{a_{ij}}{1-f_T}(u_j - x_j) \geq 1 \, .$$

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$$- \sum_{j \in L^-} \frac{a_{ij}}{f_T}(x_j - l_j) + \sum_{j \in U^+} \frac{a_{ij}}{f_T}(u_j - x_j) \geq 1 \, .$$

- Therefore these two equations imply (note that all sums-blocks are non-negative)

$$\sum_{j \in L^+} \frac{a_{ij}}{1 - f_T}(x_j - l_j) - \sum_{j \in L^-} \frac{a_{ij}}{f_T}(x_j - l_j)$$

$$+ \sum_{j \in U^+} \frac{a_{ij}}{f_T}(u_j - x_j) - \sum_{j \in U^-} \frac{a_{ij}}{1 - f_T}(u_j - x_j) \quad \geq 1 \, .$$

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$$\sum_{j\in L^+} \frac{a_{ij}}{1-f_T}(x_j - l_j) - \sum_{j\in L^-} \frac{a_{ij}}{f_T}(x_j - l_j)$$
$$+\sum_{j\in U^+} \frac{a_{ij}}{f_T}(u_j - x_j) - \sum_{j\in U^-} \frac{a_{ij}}{1-f_T}(u_j - x_j) \quad \geq 1\,.$$

- Since each of the elements on the left-hand side is equal to zero under the current assignment α, this assignment α is ruled out by the new constraint.
- In other words: the solution to the linear problem augmented with the constraint is guaranteed to be different from the previous one.