Satisfiability Checking

Branch and Bound

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Integer linear systems

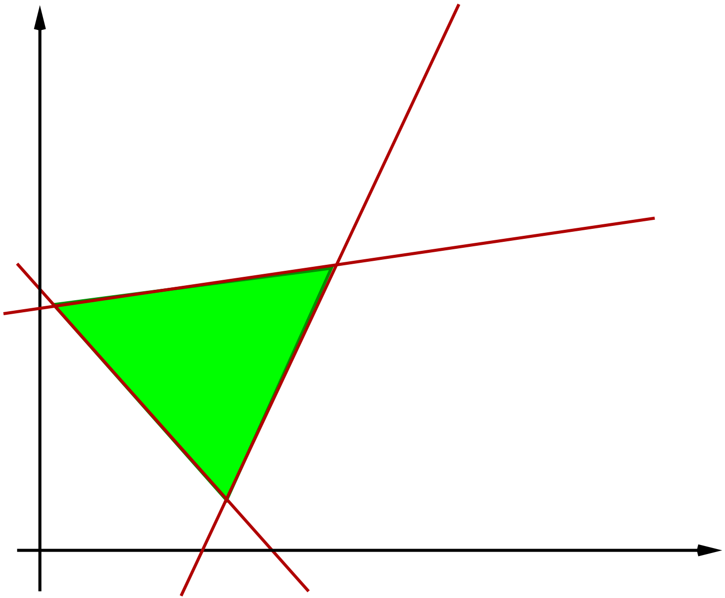
Definition

An integer linear system S is a linear system $Ax = 0$, $\bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i$, with the additional integrality requirement that all variables are of type integer.

Definition (relaxed system)

Given an integer linear system S, its relaxation relaxed(S) is S without the integrality requirement.

Geometric interpretation



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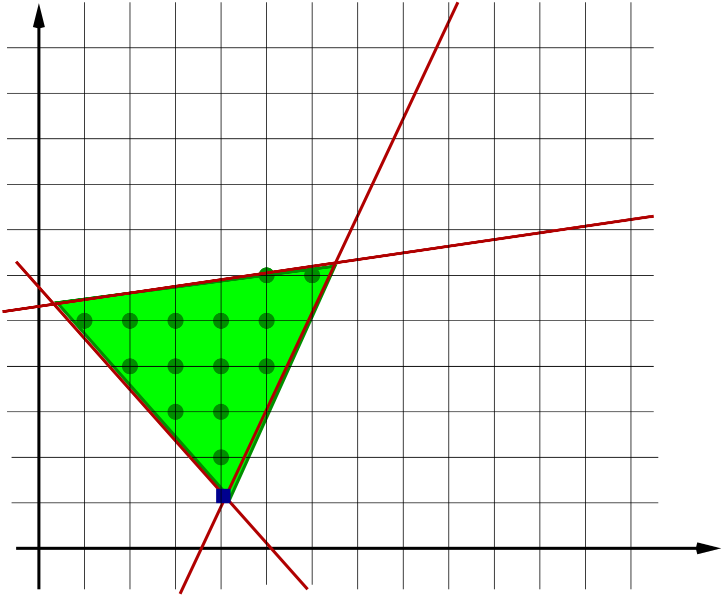

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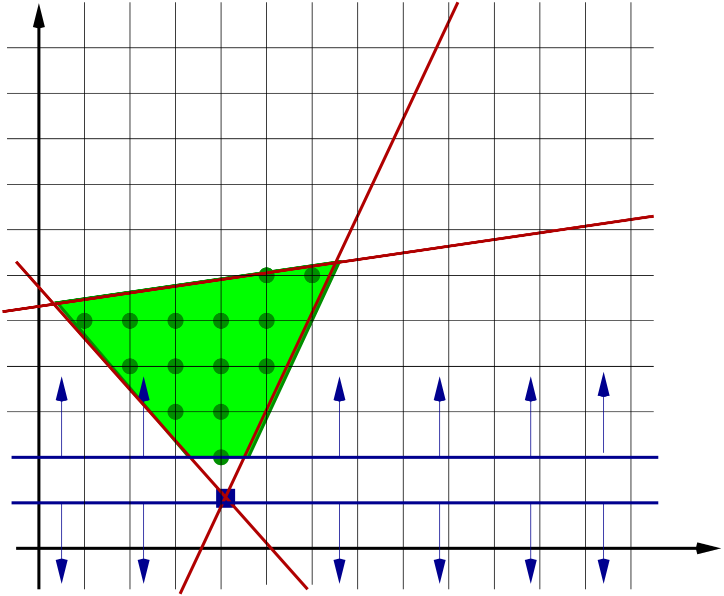

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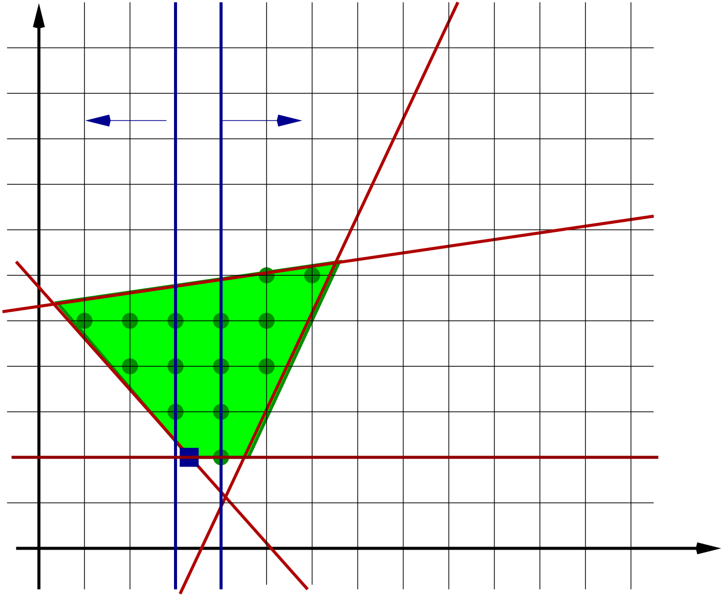

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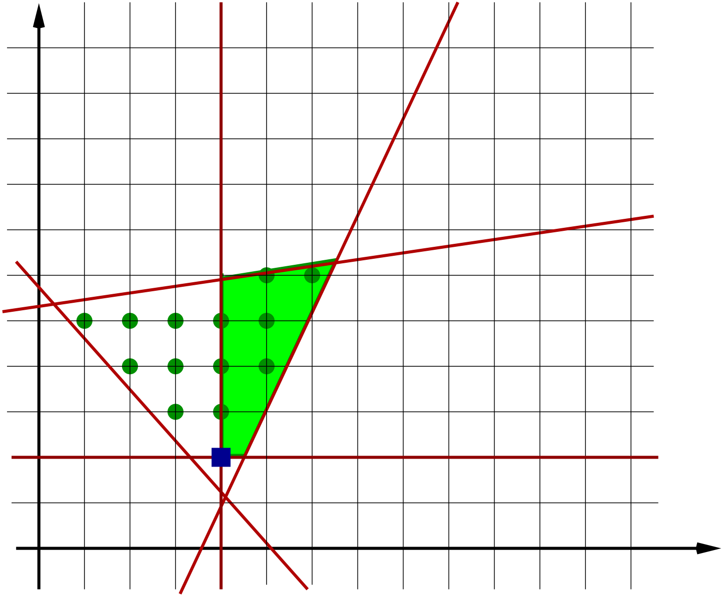

Branch and bound algorithm

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Output: SAT if S is satisfiable, UNSAT otherwise

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    if (Branch-and-Bound(S ∪ (v ≤ ⌊r⌋))==SAT) return SAT;
    else if (Branch-and-Bound(S ∪ (v ≥ ⌈r⌉))==SAT) return SAT;
    else return UNSAT;
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```

Branch and bound

- The algorithm is incomplete.
- Example: $1 \leq 3x - 3y \leq 2$ has unbounded real solutions but no integer solutions $\rightarrow$ the algorithm loops forever.
- The algorithm can be made complete for formulae with the small-model property: if there is a solution, then there is also a solution within a (computable) finite bound.
- The algorithm can be extended to mixed integer linear programming, where some of the variables are integer-valued while the others are real-valued.

Branch and bound

- Branch: Split the search space
- Bound: Exclude unsatisfiable sub-spaces
- We have seen: Depth-first search
- Also possible: Breadth-first search

Branch and bound

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  From the first two constraints we get $x_1 \leq -1$
- Assume a constraint $\sum_i a_i x_i \leq b$ with $l_i \leq x_i \leq u_i$.
  If $a_k > 0$, we have $x_k \leq (b - \sum_{i \neq k} a_i l_i)/a_k$.
  If $a_k < 0$, we have $x_k \geq (b - \sum_{i \neq k} a_i u_i)/a_k$.