Satisfiability Checking

The Simplex Algorithm

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Informatik 2
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WS 19/20

Satisfiability with Simplex

- Simplex was originally designed for solving linear programming problems, i.e., to find a solution for a set of linear real arithmetic constraints that is optimal with respect to an objective function.
- We are only interested in the feasibility problem (= satisfiability problem), which is solved in the first phase of the simplex method. In this lecture we do not handle the second phase for optimization.
- Assumption: we assume that there are no strict inequalities (simplex can be exteded to strict inequalities but it is a bit involved and we do not handle that case in the lecture)

General Simplex

- We will learn a variant called general simplex.
- Well-suited for solving the satisfiability problem fast.

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- The input is a set of constraints $\sum_{j=1}^{n} a_{ij} x_j \bowtie_i b_j$, where $a_{ij}, b_j \in \mathbb{Q}$, x_j are real-valued variables and $\bowtie_i \in \{=, \leq, \geq\}$ for m constraints $i = 1, \ldots, m$ and n variables $j = 1, \ldots, n$.

- First step: convert the input to general form

Transformation to general form

- Replace each $\sum_i a_i x_i \bowtie b_j$ (where $\bowtie \in \{=, \leq, \geq\}$) by $\sum_i a_i x_i - s_j = 0$ and $s_j \bowtie b_j$.
- Note: no $>, <$!
- $s_1, \ldots, s_m$ are called the *additional or slack variables*

Definition (General Form)

$$A(\vec{x}, \vec{s}) = 0 \quad \text{and} \quad \bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i \quad l_i \in \mathbb{Q} \cup \{-\infty\},\ u_i \in \mathbb{Q} \cup \{+\infty\}$$

Example 1

Convert $x + y \geq 2$!

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Result:
$x + y - s_1 = 0$
$s_1 \geq 2$

It is common to keep the conjunctions implicit

Example 2

Convert

$$\begin{aligned} x \quad &+y &\geq 2 \\ 2x \quad &-y &\geq 0 \\ -x \quad &+2y &\geq 1 \end{aligned}$$

Example 2

Convert

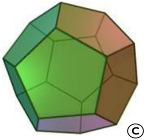
$$
\begin{array}{rrl}
x & +y & \geq 2 \\
2x & -y & \geq 0 \\
-x & +2y & \geq 1
\end{array}
$$

Result:

$$
\begin{array}{rrrl}
x & +y & -s_1 & = 0 \\
2x & -y & -s_2 & = 0 \\
-x & +2y & -s_3 & = 0 \\
 & & s_1 & \geq 2 \\
 & & s_2 & \geq 0 \\
 & & s_3 & \geq 1
\end{array}
$$

Geometrical interpretation

Linear inequality constraints, geometrically, define a convex polyhedron.



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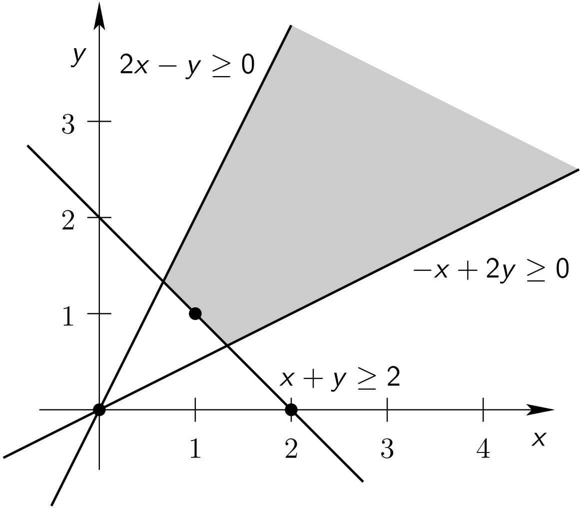
Geometrical interpretation

Our example from before:

$$\begin{array}{rrl} x & +y & \geq 2 \\ 2x & -y & \geq 0 \\ -x & +2y & \geq 0 \end{array}$$



Matrix form

- Recall the general form: $A(\vec{x}, \vec{s}) = 0$ and $\bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i$

- A is now an $m \times (n+m)$ matrix
  due to the additional variables.

$$
\begin{array}{rrrl}
x & +y & -s_1 & = 0 \\
2x & -y & -s_2 & = 0 \\
-x & +2y & -s_3 & = 0 \\
 & & s_1 & \geq 2 \\
 & & s_2 & \geq 0 \\
 & & s_3 & \geq 1
\end{array}
\qquad
\begin{array}{c}
\begin{array}{rrrrr} x & y & s_1 & s_2 & s_3 \end{array} \\
\begin{pmatrix}
1 & 1 & -1 & 0 & 0 \\
2 & -1 & 0 & -1 & 0 \\
-1 & 2 & 0 & 0 & -1
\end{pmatrix}
\end{array}
$$

The tableau

- The diagonal part is inherent to the general form:

$$\begin{array}{ccccc} x & y & s_1 & s_2 & s_3 \end{array}$$
$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 & -1 \end{pmatrix}$$

- Instead, we can write:

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \end{pmatrix} \end{array}$$

The tableau

- The tableaux changes throughout the algorithm, but maintains its $m \times n$ structure
- Distinguish basic (also called dependent) and non-basic variables

$$
\begin{array}{cc}
 & \begin{array}{cc} x & y \end{array} \leftarrow \text{Non-basic variables} \\
\text{Basic variables} \rightarrow \begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} & \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \end{pmatrix}
\end{array}
$$

Notation:

$\mathcal{B}$	the set of basic variables
$\mathcal{N}$	the set of non-basic variables

- Initially, basic variables = the additional variables
- The tableaux is simply a different notation for the system

$$\bigwedge_{s_i \in \mathcal{B}} \left(s_i = \sum_{x_j \in \mathcal{N}} a_{ij} x_j \right)$$

- The basic variables are also called the dependent variables.

Data structures

- Simplex maintains:
  - The tableau,
  - an assignment α to all (problem and additional) variables.
- Initially, $\alpha(x_i) = 0$ for $i \in \{1, ..., n+m\}$

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  1. $A\vec{x} = 0$
  2. All non-basic variables satisfy their bounds
- The basic variables do not need to satisfy their bounds.

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- The basic variables do not need to satisfy their bounds.

- Can you see why these invariants are maintained initially?

Invariants

- The initial assignment satisfies $A\vec{x} = 0$

- If the bounds of all basic variables are satisfied by α, return "satisfiable".

- Otherwise... *pivot*.

Pivoting

1. Find a basic variable x_i that violates its bounds. Assume $\alpha(x_i) < l_i$.
2. Find a non-basic variable x_j such that
   - $a_{ij} > 0$ and $\alpha(x_j) < u_j$, or
   - $a_{ij} < 0$ and $\alpha(x_j) > l_j$.

   Why?

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   Why? If there is such an x_j then we can increase the value of x_i to its lower bound, and in compensation we can change the value of x_j to maintain the equation in the row of x_i. Such a variable is called suitable. However, if a_{ij} would be 0 then we cannot compensate the value change of x_i by changing the value of x_j. If $a_{ij} > 0$ and $\alpha(x_j) \geq u_j$ then x_j is on its upper bound (remember that non-basic variables satisfy their bounds) and we need to further increase its value in order to compensate the value increment for x_i, however, then we would need to decrement it later with at least the same value. The case for $a_{ij} < 0$ and $\alpha(x_j) \leq l_j$ is similar.

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3. If there is no suitable variable, return "unsatisfiable".

   Why? The maximal value of the linear term to which x_i should be equal to is smaller than the lower bound on x_i.

Pivoting x_i and x_j (1)

1. Solve equation i for x_j:

$$\text{From:} \quad x_i = a_{ij}x_j + \sum_{k \neq j} a_{ik}x_k$$

$$\text{To:} \quad x_j = \frac{x_i}{a_{ij}} - \sum_{k \neq j} \frac{a_{ik}}{a_{ij}} x_k$$

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2. Swap x_i and x_j, and update the i-th row accordingly

From:

a_{i1}	$\dots$	a_{ij}	$\dots$	a_{in}

To:

$\frac{-a_{i1}}{a_{ij}}$	$\dots$	$\frac{1}{a_{ij}}$	$\dots$	$\frac{-a_{in}}{a_{ij}}$

Pivoting x_i and x_j (2)

3. Update all other rows:
   Replace x_j with its equivalent obtained from row i:

$$x_j = \frac{x_i}{a_{ij}} - \sum_{k \neq j} \frac{a_{ik}}{a_{ij}} x_k$$

Pivoting x_i and x_j (2)

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$$x_j = \frac{x_i}{a_{ij}} - \sum_{k \neq j} \frac{a_{ik}}{a_{ij}} x_k$$

4. Update α as follows:
   - Increase $\alpha(x_j)$ by $\theta = \dfrac{l_i - \alpha(x_i)}{a_{ij}}$
     Now x_j is a basic variable: it may violate its bounds.
     Update $\alpha(x_i)$ accordingly.
     Q: What is $\alpha(x_i)$ now?
   - Update α for all other basic (dependent) variables.

Pivoting: Example (1)

- Recall the tableau and constraints in our example:

	x	y
s_1	1	1
s_2	2	-1
s_3	-1	2

$$\begin{array}{rcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

- Initially,

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- We will fix s_1.
- x is a *suitable* non-basic variable for pivoting.
  It has no upper bound!
- So now we pivot s_1 with x

Pivoting: Example (2)

	x	y
s_1	1	1
s_2	2	-1
s_3	-1	2

$$
\begin{aligned}
2 &\leq s_1 \\
0 &\leq s_2 \\
1 &\leq s_3
\end{aligned}
$$

Pivoting: Example (2)

	x	y
s_1	1	1
s_2	2	-1
s_3	-1	2

$$2 \leq s_1$$
$$0 \leq s_2$$
$$1 \leq s_3$$

- Solve 1st row for x:

$$s_1 = x + y \quad \leftrightarrow \quad x = s_1 - y$$

Pivoting: Example (2)

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$$
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0 &\leq s_2 \\
1 &\leq s_3
\end{aligned}
$$

- Solve 1$^{\text{st}}$ row for x:

$$s_1 = x + y \quad \leftrightarrow \quad x = s_1 - y$$

- Replace x in other rows:

$$
\begin{aligned}
s_2 &= 2(s_1 - y) - y & \leftrightarrow \quad s_2 &= 2s_1 - 3y \\
s_3 &= -(s_1 - y) + 2y & \leftrightarrow \quad s_3 &= -s_1 + 3y
\end{aligned}
$$

Pivoting: Example (3)

$$\begin{aligned} x &= s_1 - y \\ s_2 &= 2s_1 - 3y \\ s_3 &= -s_1 + 3y \end{aligned}$$

Pivoting: Example (3)

This results in the following new tableau:

$$\begin{array}{lcl} x & = & s_1 - y \\ s_2 & = & 2s_1 - 3y \\ s_3 & = & -s_1 + 3y \end{array}$$

	s_1	y
x	1	-1
s_2	2	-3
s_3	-1	3

$$\begin{array}{lcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

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What about the assignment?

Pivoting: Example (3)

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	s_1	y
x	1	-1
s_2	2	-3
s_3	-1	3

$$\begin{array}{rcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

What about the assignment?

- We should increase x by $\theta = \frac{2-0}{1} = 2$
- Hence, $\alpha(x) = 0 + 2 = 2$
- Now s_1 is equal to its lower bound: $\alpha(s_1) = 2$
- Update all the others

Pivoting: Example (4)

The new state:

	s_1	y
x	1	-1
s_2	2	-3
s_3	-1	3

$$\begin{aligned}\alpha(x) &= 2\\ \alpha(y) &= 0\\ \alpha(s_1) &= 2\\ \alpha(s_2) &= 4\\ \alpha(s_3) &= -2\end{aligned}$$

$$\begin{aligned}2 &\leq s_1\\ 0 &\leq s_2\\ 1 &\leq s_3\end{aligned}$$

Pivoting: Example (4)

The new state:

	s_1	y
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$$\begin{array}{lcr} \alpha(x) & = & 2 \\ \alpha(y) & = & 0 \\ \alpha(s_1) & = & 2 \\ \alpha(s_2) & = & 4 \\ \alpha(s_3) & = & -2 \end{array}$$

$$\begin{array}{rcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

- Now s_3 violates its lower bound
- Which non-basic variable is suitable for pivoting?

Pivoting: Example (4)

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$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

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  That's right… y

Pivoting: Example (4)

The new state:

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$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

- Now s_3 violates its lower bound
- Which non-basic variable is suitable for pivoting?
  That's right... y
- We should increase y by $\theta = \frac{1-(-2)}{3} = 1$

Pivoting: Example (5)

The final state:

	s_1	s_3
x	$2/3$	$-1/3$
s_2	1	-1
y	$1/3$	$1/3$

$$\begin{aligned} \alpha(x) &= 1 \\ \alpha(y) &= 1 \\ \alpha(s_1) &= 2 \\ \alpha(s_2) &= 1 \\ \alpha(s_3) &= 1 \end{aligned}$$

$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

All constraints are satisfied.

Observations I

The additional variables:

- Only additional variables have bounds.
- These bounds are permanent.
- Additional variables enter the base only on extreme points (their lower or upper bounds).
- When entering the base, they shift towards the other bound and possibly cross it (violate it).

Observations II

Q: Can it be that we pivot x_i, x_j and then pivot x_j, x_i
and thus enter a (local) cycle?

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A: No.

- For example, suppose that $a_{ij} > 0$.
- We increased $\alpha(x_j)$ so now $\alpha(x_i) = l_i$.
- After pivoting, possibly $\alpha(x_j) > u_j$, but $a'_{ij} = 1/a_{ij} > 0$, hence the coefficient of x_i is not suitable

Termination

Is termination guaranteed?

Termination

Is termination guaranteed?

- Not obvious. Perhaps there are bigger cycles.
- In order to avoid circles, we use Bland's rule:
  1. Determine a total order on the variables
  2. Choose the first basic variable that violates its bounds, and the first non-basic suitable variable for pivoting.
  3. It can be shown that this guarantees that no base is repeated, which implies termination.

General simplex with Bland's rule

1. Transform the system into the general form

$$A(\vec{x}, \vec{s}) = 0 \quad \text{and} \quad \bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i \ .$$

2. Set $\mathcal{B}$ to be the set of additional variables $s_1, \ldots, s_m$.
3. Construct the tableau for A.
4. Determine a fixed order on the variables.
5. If there is no basic variable that violates its bounds, return "satisfiable". Otherwise, let x_i be the first basic variable in the order that violates its bounds.
6. Search for the first suitable non-basic variable x_j in the order for pivoting with x_i. If there is no such variable, return "unsatisfiable".
7. Perform the pivot operation on x_i and x_j.
8. Go to step 5.