

Satisfiability Checking

Lazy SAT-Modulo-Theories (SMT) Solving

Prof. Dr. Erika Ábrahám

RWTH Aachen University
Informatik 2
LuFG Theory of Hybrid Systems

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The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

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Logic:

The Xmas problem

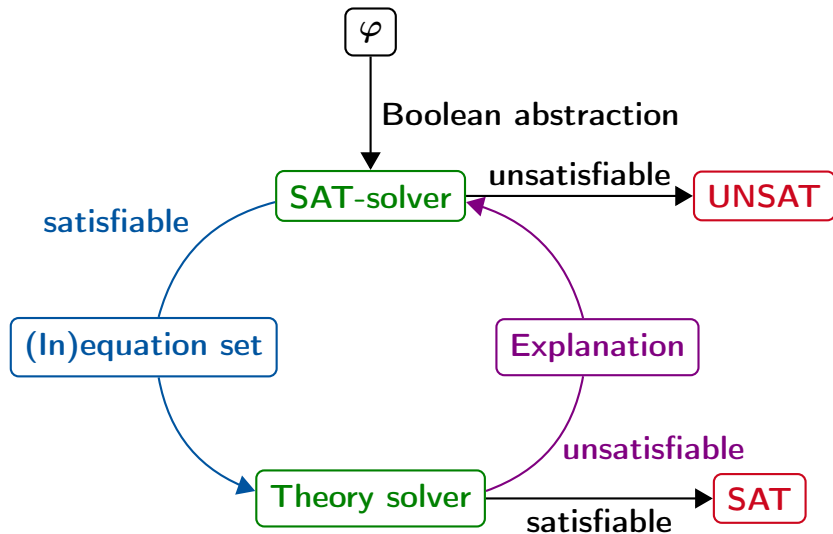
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Logic: First-order logic over the integers with addition.

Full lazy SMT-solving



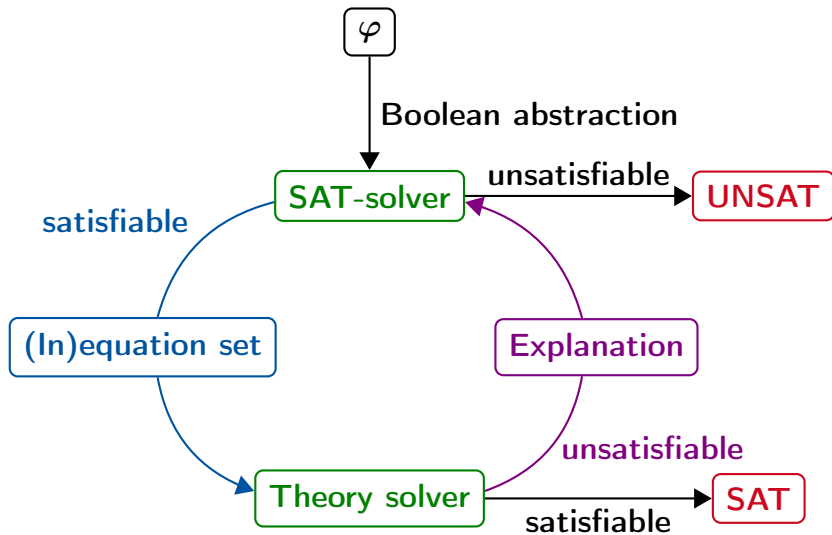
Boolean abstraction

$$\underbrace{(p_1 = 0)}_{a_1} \vee \underbrace{(p_2 = 0)}_{a_2} \vee \underbrace{(p_3 = 0)}_{a_3} \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$\underbrace{(p_1 \geq 5)}_{a_5} \vee \underbrace{(p_2 \geq 5)}_{a_6} \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
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$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Full lazy SMT-solving



$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

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DL0 :

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Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : a_4 : 1

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

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DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 :

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Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 : $a_1 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

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DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

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DL2 :

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

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DL3 :

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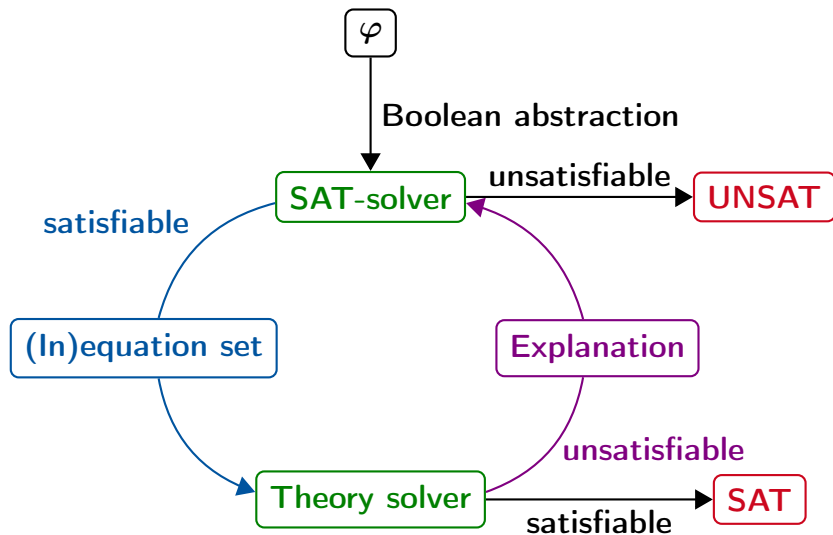
DL1 : $a_1 : 0$

DL2 : $a_2 : 0, a_3 : 1$

DL3 : $a_5 : 0, a_6 : 1$

Solution found for the Boolean abstraction.

Full lazy SMT-solving



Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $DL1 : a_1 : 0$
 $DL2 : a_2 : 0, a_3 : 1$ $DL3 : a_5 : 0, a_6 : 1$

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True theory constraints: $a_4, a_7, a_8, a_9, a_3, a_6$

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$$\underbrace{(p_1 = 0)}_{a_1} \vee \underbrace{(p_2 = 0)}_{a_2} \vee \underbrace{(p_3 = 0)}_{a_3} \wedge \underbrace{(p_1 + p_2 + p_3 \geq 100)}_{a_4} \wedge$$
$$\underbrace{(p_1 \geq 5)}_{a_5} \vee \underbrace{(p_2 \geq 5)}_{a_6} \wedge \underbrace{(p_3 \geq 10)}_{a_7} \wedge \underbrace{(p_1 + 2p_2 + 5p_3 \leq 180)}_{a_8} \wedge$$
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Encoding:

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_3 : p_3 = 0$

$a_6 : p_2 \geq 5$

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

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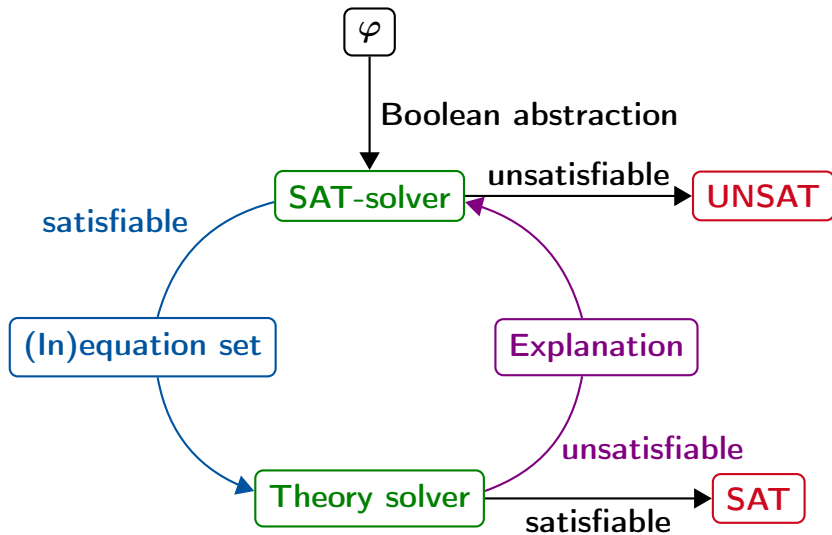
$$a_3 : p_3 = 0$$

$$a_6 : p_2 \geq 5$$

No.

Reason: $\underbrace{p_3 = 0}_{a_3} \wedge \underbrace{p_3 \geq 10}_{a_7}$ are conflicting.

Full lazy SMT-solving



Add clause $(\neg a_3 \vee \neg a_7)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

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DL2 : $a_2 : 0, a_3 : 1$

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Conflict resolution is simple, since the new clause is already an asserting one.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

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DL2 :

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DL1 : $a_1 : 0, a_2 : 1$

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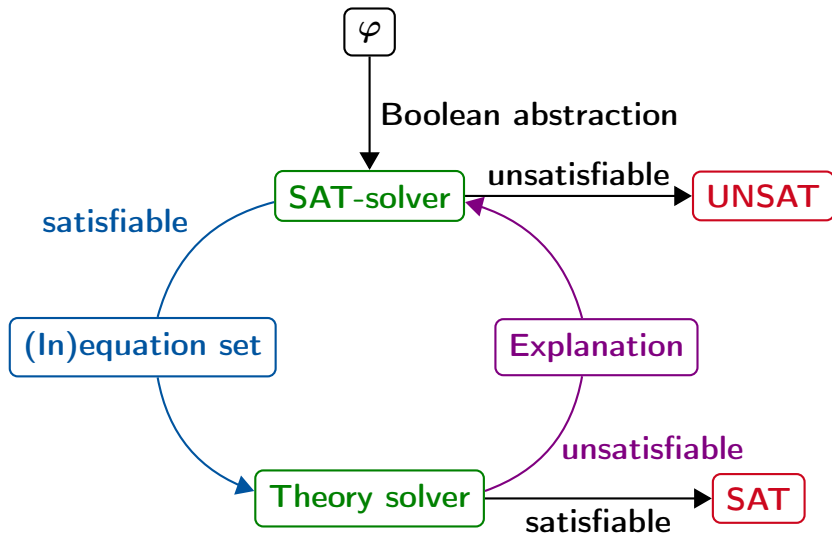
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Solution found for the Boolean abstraction.

Full lazy SMT-solving



$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1$

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Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1$

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True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

$$\begin{aligned} & (\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge \\ & (\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge \\ & \quad \underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \end{aligned}$$

Theory solving

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True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

$$\underbrace{(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0)}_{a_1} \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
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Encoding:

$$a_4 : p_1 + p_2 + p_3 \geq 100 \quad a_7 : p_3 \geq 10 \quad a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$
$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300 \quad a_2 : p_2 = 0 \quad a_6 : p_2 \geq 5$$

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

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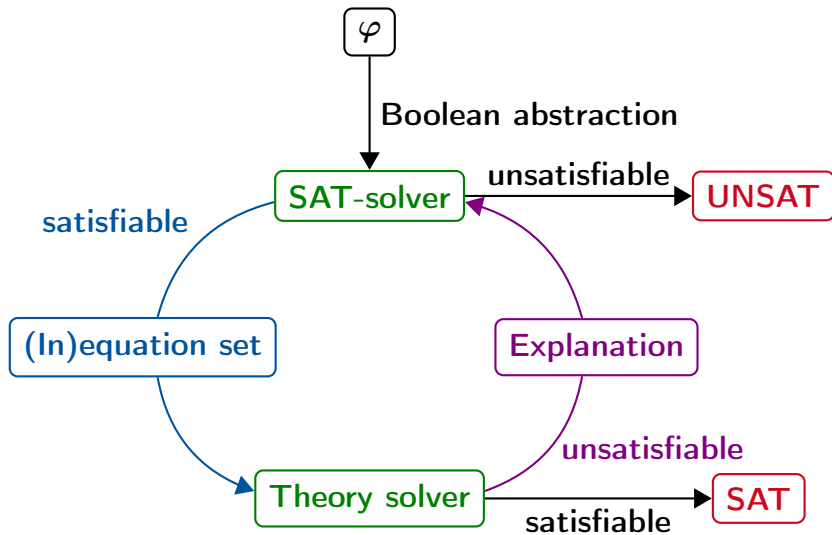
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Reason: $\underbrace{p_2 = 0}_{a_2} \wedge \underbrace{p_2 \geq 5}_{a_6}$ are conflicting.

Full lazy SMT-solving



Add clause $(\neg a_2 \vee \neg a_6)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1$

DL2 : $a_5 : 0, a_6 : 1$

Add clause $(\neg a_2 \vee \neg a_6)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1$

DL2 : $a_5 : 0, a_6 : 1$

Conflict resolution is simple, since the new clause is already an asserting one.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
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DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

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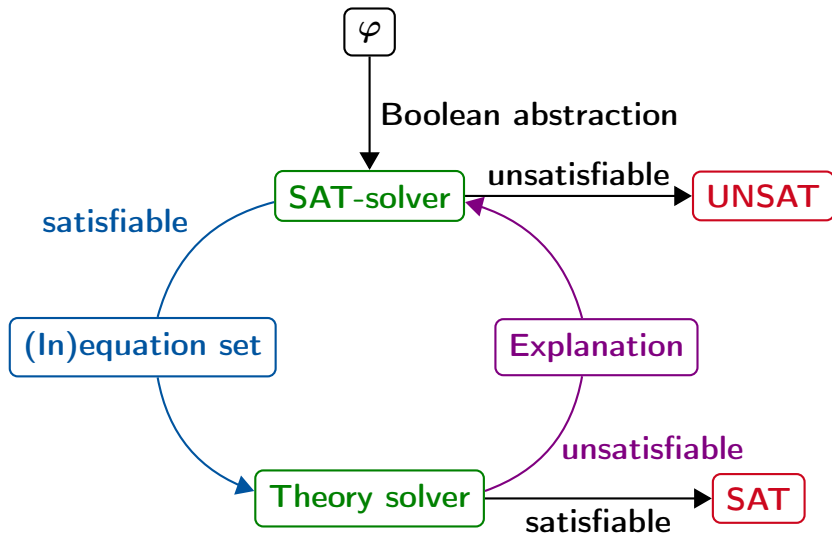
$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

Solution found for the Boolean abstraction.

Full lazy SMT-solving



$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$$\underbrace{(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0)}_{a_1} \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$\underbrace{(p_1 \geq 5 \vee p_2 \geq 5)}_{a_5} \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$$\underbrace{(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0)}_{a_1} \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$\underbrace{(p_1 \geq 5 \vee p_2 \geq 5)}_{a_5} \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

Encoding:

$$a_4 : p_1 + p_2 + p_3 \geq 100 \quad a_7 : p_3 \geq 10 \quad a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$
$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300 \quad a_2 : p_2 = 0 \quad a_5 : p_1 \geq 5$$

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_2 : p_2 = 0$$

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$$a_2 : p_2 = 0$$

$$a_5 : p_1 \geq 5$$

Yes.

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_2 : p_2 = 0$$

$$a_5 : p_1 \geq 5$$

Yes. E.g.,

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

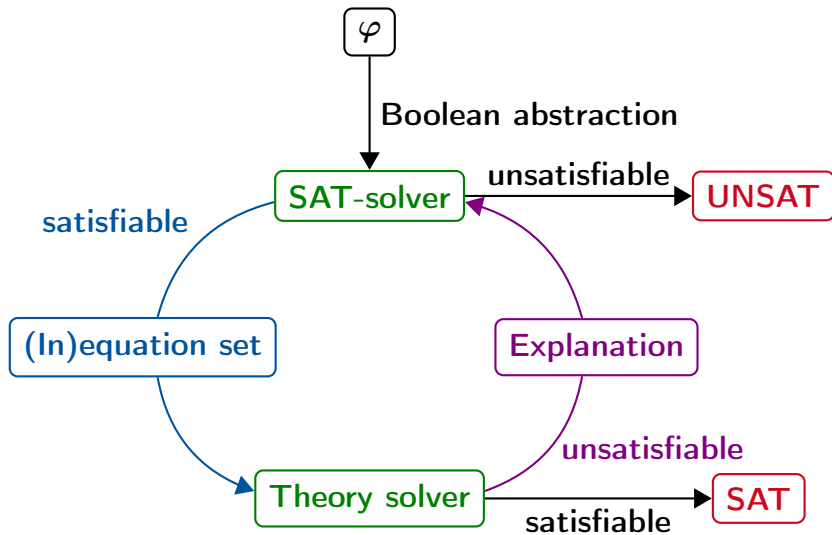
$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_2 : p_2 = 0$$

$$a_5 : p_1 \geq 5$$

Yes. E.g., $p_1 = 90$, $p_2 = 0$, $p_3 = 10$ is a solution.

Full lazy SMT-solving



Full lazy SMT-solving

Input: Quantifier-free FO logic formula φ over some theories in CNF without any negation

Output: Satisfiability of the input formula

- Let C be the set of all theory constraints in φ
- Let $P = \{p_c | c \in C\}$ be a set of fresh atomic propositions (fresh means not appearing in φ)
- Let $\mu : C \rightarrow P$ be the bijective function with $\mu(c) = p_c$ and $\mu^{-1}(p_c) = c$
- For each formula φ' with constraints from C we define the Boolean abstraction (or Boolean skeleton) $\mu(\varphi')$ of φ' under μ to be the propositional logic formula we get by replacing each theory constraint c in φ' by $\mu(c)$

Full lazy SMT-solving

Input: Quantifier-free FO logic formula φ over some theories in CNF without any negation

Output: Satisfiability of the input formula

- 1 Build the Boolean skeleton (also called Boolean abstraction) φ_{abs} of the input formula φ by replacing each theory constraint $c \in C$ in φ by $\mu(c)$
- 2 Search for a solution for φ_{abs} (using SAT solving)
- 3 If there is no solution for φ_{abs} then the input formula φ is unsatisfiable
- 4 Otherwise, given a solution $\alpha : P \rightarrow \{0, 1\}$ for φ_{abs} , check the set of all true theory constraints $C_\mu := \{c \in C \mid \alpha(\mu(c)) = 1\}$ for consistency
- 5 If they are consistent then input formula is satisfiable
- 6 Otherwise, compute an explanation for the inconsistency in form of CNF formula with constraints from C implying that the constraints in C_μ cannot be all true
- 7 Learn the Boolean abstraction E of the theory lemma by setting $\varphi_{abs} := \varphi_{abs} \wedge E$
- 8 Apply conflict resolution if the learnt clause is not asserting
- 9 Goto 2

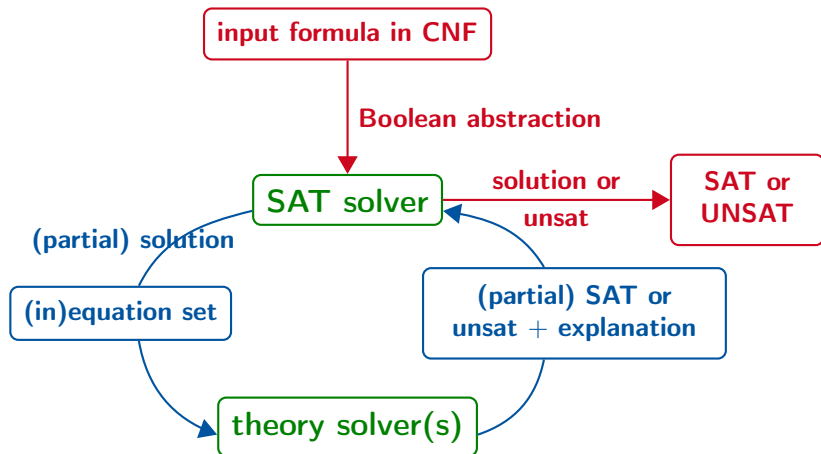
Full lazy SMT-solving

Input: Quantifier-free FO logic formula φ over some theories in CNF **without any negation**

Output: Satisfiability of the input formula

- 1 Build the Boolean skeleton (also called Boolean abstraction) φ_{abs} of the input formula φ by replacing each theory constraint $c \in C$ in φ by $\mu(c)$
- 2 Search for a solution for φ_{abs} (using SAT solving)
- 3 If there is no solution for φ_{abs} then the input formula φ is unsatisfiable
- 4 Otherwise, given a solution $\alpha : P \rightarrow \{0, 1\}$ for φ_{abs} , check the **set of all true theory constraints** $C_\mu := \{c \in C \mid \alpha(\mu(c)) = 1\}$ for consistency
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- 9 Goto 2

Less lazy SMT-solving



- 1 **Incrementality**: In less lazy solving we extend the set of constraints. The solver should make use of the previous satisfiability check for the check of the extended set.
- 2 **(Preferably minimal) infeasible subsets**: Compute a reason for unsatisfaction
- 3 **Backtracking**: The theory solver should be able to remove constraints in inverse chronological order.

- This approach strictly divides between logical (Boolean) structure and theory constraints.
- There are other approaches, which do not divide Boolean and theory solving so strictly.
- One idea: Propagate in the SAT-solver **bounds** on theory variables.