Satisfiability Checking

Propositional Logic

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Informatik 2
LuFG Theory of Hybrid Systems

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Propositional logic

The slides are partly taken from:

www.decision-procedures.org/slides/

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Modeling with propositional logic
- Normal forms
- Enumeration and deduction

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- We omit parentheses whenever we may restore them through operator precedence:

binds stronger

$\longleftarrow$

$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$

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Structures for predicate logic:

- The domain is $\mathbb{B} = \{0, 1\}$.
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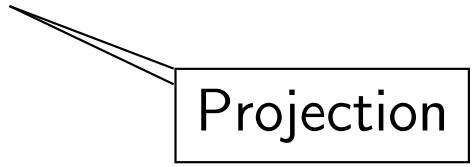
Only the projected assignment matters...

- Let $\alpha_1, \alpha_2 \in$ *Assign* and $\varphi \in$ *PropForm*.
- Let $AP(\varphi)$ be the atomic propositions in φ.
- Clearly $AP(\varphi) \subseteq AP$.
- Lemma: if $\alpha_1|_{AP(\varphi)} = \alpha_2|_{AP(\varphi)}$, then



$$(\alpha_1 \textit{ satisfies } \varphi) \quad \text{iff} \quad (\alpha_2 \textit{ satisfies } \varphi)$$

- We will assume, for simplicity, that $AP = AP(\varphi)$.

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α satisfies φ iff in the line for α and the column for φ the entry is 1.

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- Q: Does α satisfy φ?
- A1: Compute with truth table:

a	b	c	$b \rightarrow c$	$a \vee (b \rightarrow c)$
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Satisfaction relation: $\models \subseteq \mathit{Assign} \times \mathit{PropForm}$

Instead of $(\alpha, \varphi) \in \models$ we write $\alpha \models \varphi$ and say that

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\alpha \models p & \textit{iff} & \alpha(p) = \textit{true} \\
\alpha \models \neg\varphi & \textit{iff} & \alpha \not\models \varphi \\
\alpha \models \varphi_1 \wedge \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ and } \alpha \models \varphi_2 \\
\alpha \models \varphi_1 \vee \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ or } \alpha \models \varphi_2 \\
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\alpha \models \varphi_1 \leftrightarrow \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ iff } \alpha \models \varphi_2
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Note: More elegant but semantically equivalent to truth tables.

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A2: Compute with the satisfaction relation:

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- Using the satisfaction relation we can define an algorithm for the problem to decide whether an assignment $\alpha : AP \to \{0, 1\}$ is a model of a propositional logic formula $\varphi \in \mathit{PropForm}$:

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- Hence, $\alpha \models \varphi$.

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- For $\varphi \in PropForm$ and $\alpha \in Assign$ it holds that

$$\alpha \models \varphi \quad \textit{iff} \quad \alpha \in sat(\varphi)$$

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Extensions of $\models$

- We define $\models \subseteq 2^{Assign} \times PropForm$ by

$$T \models \varphi \text{ iff } T \subseteq sat(\varphi)$$

for formulae $\varphi \in PropForm$ and assignment sets $T \subseteq 2^{Assign}$.

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Examples: $\{\alpha \in Assign \mid \alpha(a) = \alpha(c) = 1\} \models a \vee (b \rightarrow c)$

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Examples: $a \wedge c \models a \vee (b \rightarrow c)$
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Short summary for propositional logic

- Syntax of propositional formulae $\varphi \in \mathit{PropForm}$:

$$\varphi \ := \ AP \mid (\neg\varphi) \mid (\varphi \wedge \varphi)$$

- Semantics:
  - Assignments $\alpha \in \mathit{Assign}$:

$$\alpha : AP \to \{0,1\}$$
$$\alpha \in 2^{AP}$$
$$\alpha \in \{0,1\}^{AP}$$

  - Satisfaction relation:

$$\begin{array}{lll} \models \subseteq \mathit{Assign} \times \mathit{PropForm} & , & (\text{e.g., } \alpha \models \varphi) \\ \models \subseteq 2^{\mathit{Assign}} \times \mathit{PropForm} & , & (\text{e.g., } \{\alpha_1, \ldots, \alpha_n\} \models \varphi) \\ \models \subseteq \mathit{PropForm} \times \mathit{PropForm}, & & (\text{e.g., } \varphi_1 \models \varphi_2) \\ \mathit{sat} : \ \mathit{PropForm} \to 2^{\mathit{Assign}} & , & (\text{e.g., } \ \mathit{sat}(\varphi)) \end{array}$$

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Modeling with propositional logic
- Normal forms
- Enumeration and deduction

Semantic classification of formulae

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  (Also called a tautology).

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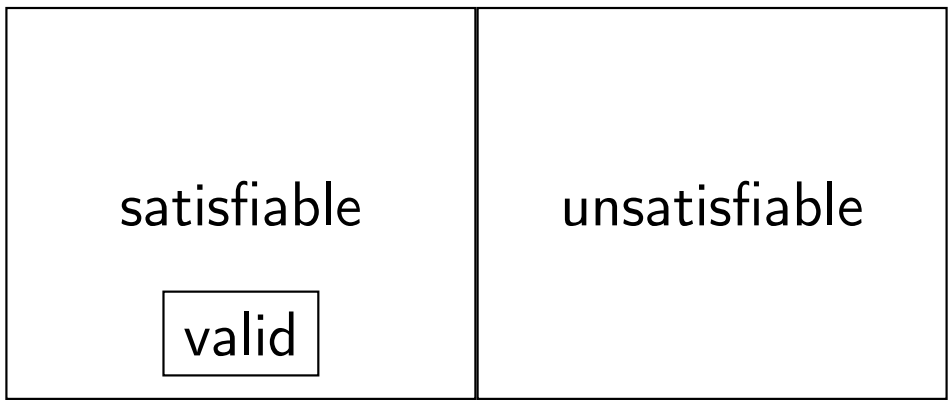

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- $(x_1 \vee x_2) \rightarrow x_1$ is satisfiable
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  - $\models a \wedge (b \vee c) \leftrightarrow (a \wedge b) \vee (a \wedge c)$
- Some more (De Morgan rules):
  - $\models \neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b)$
  - $\models \neg(a \vee b) \leftrightarrow (\neg a \wedge \neg b)$

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  Given an input propositional formula φ, decide whether φ is satisfiable.

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- This problem is decidable but NP-complete.

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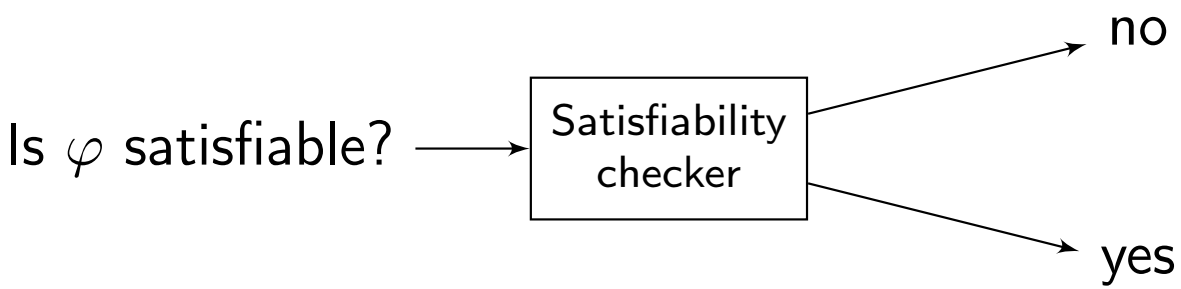

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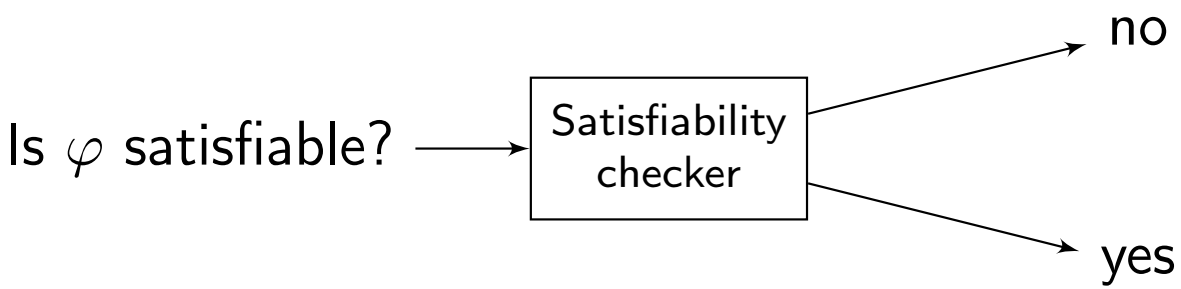

Note: A formula φ is valid iff $\neg\varphi$ is unsatisfiable.

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Modeling with propositional logic
- Normal forms
- Enumeration and deduction

Before we solve this problem...

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- There are numerous problems in the industry that are solved via the satisfiability problem of propositional logic
  - Logistics
  - Planning
  - Electronic Design Automation industry
  - Cryptography
  - ...

Example 1: Placement of wedding guests

- Three chairs in a row: $1, 2, 3$
- We need to place Aunt, Sister and Father.
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  - Aunt doesn't want to sit near Father
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- Q: Can we satisfy these constraints?

Example 1 (continued)

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- Q: Can we assign to each station a frequency, such that no station pairs from E have the same frequency?

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- n participants
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- $\varphi_1 = \neg(a \vee \neg b)$ is not in NNF
- $\varphi_2 = \neg a \wedge b$ is in NNF

Converting to NNF

- Every formula can be converted to NNF in linear time:
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- DNF is a special case of NNF.

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  A: 2^n

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  A: Linear (time and space).

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$$(a_1 \wedge a_2 \wedge \neg a_1) \vee (a_2 \wedge a_1) \vee (a_2 \wedge \neg a_3 \wedge a_3)$$

  A: Yes, because the term $a_2 \wedge a_1$ is satisfiable.
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- A: No, it would violate the NP-completeness of the problem.

Conjunctive Normal Form (CNF)

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- Also CNF is a special case of NNF.

Converting to CNF

- Every formula can be converted to CNF in exponential time and space:
  1. Convert to NNF
  2. Distribute disjunctions following the rule:
     $\models \varphi_1 \vee (\varphi_2 \wedge \varphi_3) \leftrightarrow (\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \varphi_3)$

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- Consider the formula $\varphi = (a_1 \wedge b_1) \vee (a_2 \wedge b_2)$.

  Transformation: $(a_1 \vee a_2) \wedge (a_1 \vee b_2) \wedge (b_1 \vee a_2) \wedge (b_1 \vee b_2)$

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- Now consider $\varphi_n = (a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee \ldots \vee (a_n \wedge b_n)$.

  Q: How many clauses does the resulting CNF have?

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  Q: How many clauses does the resulting CNF have?

  A: 2^n

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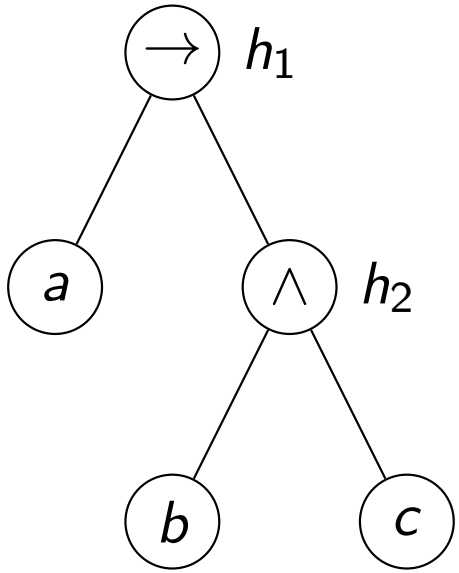
Parse tree:



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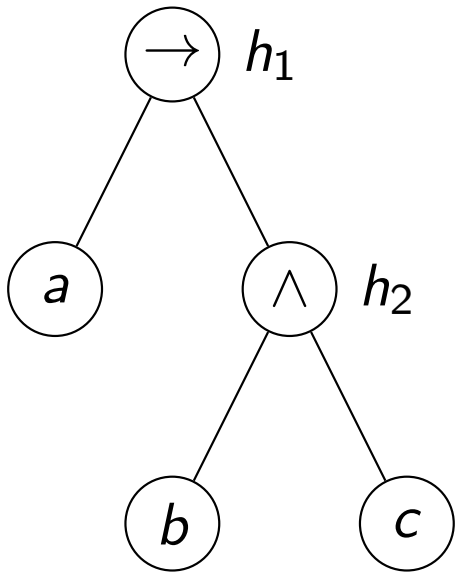

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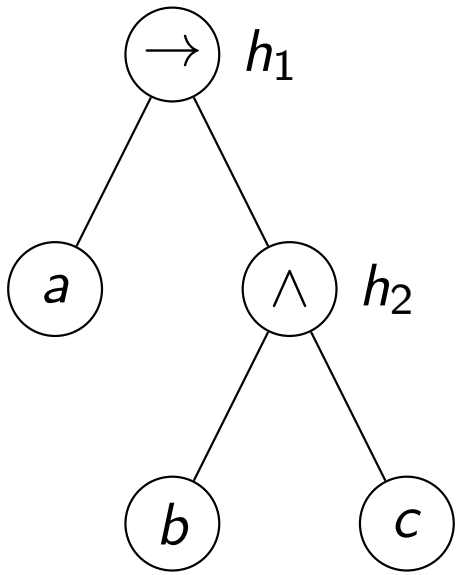

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  $\varphi = (a \rightarrow (b \wedge c))$
- Associate a new auxiliary variable with each gate.
- Add constraints that define these new variables.
- Finally, enforce the root node.

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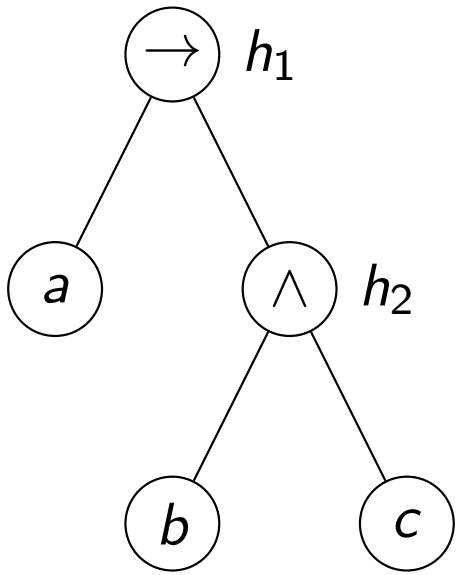


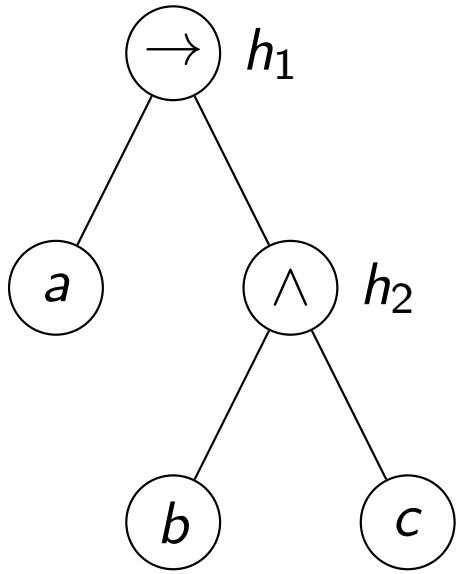
Converting to CNF: Tseitin's encoding

- Need to satisfy:

  $(h_1 \leftrightarrow (a \rightarrow h_2)) \wedge$

  $(h_2 \leftrightarrow (b \wedge c)) \wedge$

  (h_1)



  - Each gate encoding has a CNF representation with 3 or 4 clauses.

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- Second: $(\neg h_2 \vee b) \wedge (\neg h_2 \vee c) \wedge (h_2 \vee \neg b \vee \neg c)$

Converting to CNF: Tseitin's encoding

- Let's go back to

  $\varphi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \cdots \vee (x_n \wedge y_n)$

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- With Tseitin's encoding we need:
  - n auxiliary variables $h_1, \ldots, h_n$.
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- Hence, we have
  - $3n + 1$ clauses, instead of 2^n.
  - $3n$ variables rather than $2n$.

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Modeling with propositional logic
- Normal forms
- Enumeration and deduction

Two classes of algorithms for validity

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- More generally (beyond propositional logic):

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- Two classes of algorithms for finding out:
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- More generally (beyond propositional logic):
  - Enumeration is possible only in some logics.
  - Deduction cannot necessarily be fully automated.

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- Q: What is the difference?
  A: Branching on complete vs. partial assignments.

Deduction requires axioms and inference rules

- Inference rules:

$$\frac{\textit{Antecedents}}{\textit{Consequents}} \qquad (\textit{rule name})$$

Meaning: If all antecedents hold then at least one of the consequents can be derived.

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- Examples:

$$\frac{a \to b \qquad b \to c}{a \to c} \qquad (\textit{Trans})$$

$$\frac{a \to b \qquad a}{b} \qquad (\textit{M.P.})$$

Axioms

- Axioms are inference rules with no antecedents, e.g.,

$$\frac{}{a \rightarrow (b \rightarrow a)} \quad (H1)$$

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- A proof system consists of a set of axioms and inference rules.

Proofs

- Let $\mathcal{H}$ be a proof system.
- $\Gamma \vdash_{\mathcal{H}} \varphi$ means: There is a proof of φ in system $\mathcal{H}$ whose premises are included in Γ
- $\vdash_{\mathcal{H}}$ is called the provability (derivability) relation.

Example

- Let $\mathcal{H}$ be the proof system comprised of the rules Trans and M.P. that we saw earlier:

$$\frac{a \to b \quad b \to c}{a \to c} \qquad (Trans)$$

$$\frac{a \to b \quad a}{b} \qquad (M.P.)$$

- Does the following relation hold?

$$a \to b,\ b \to c,\ c \to d,\ d \to e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

Deductive proof: Example

$$\frac{a \to b \quad b \to c}{a \to c}\ (\mathit{Trans}) \qquad \frac{a \to b \quad a}{b}\ (\mathit{M.P.})$$

$$a \to b,\ b \to c,\ c \to d,\ d \to e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

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1. $a \to b$

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$a \to b,\ b \to c,\ c \to d,\ d \to e,\ a \quad \vdash_{\mathcal{H}} \quad e$

1. $a \to b$ *premise*

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3. $a \to c$ 1, 2, *Trans*

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2. $b \to c$ *premise*
3. $a \to c$ 1, 2, *Trans*
4. $c \to d$

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5. $d \to e$ *premise*
6. $c \to e$ 4, 5, *Trans*

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7. $a \to e$

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6. $c \to e$ 4, 5, *Trans*
7. $a \to e$ 3, 6, *Trans*

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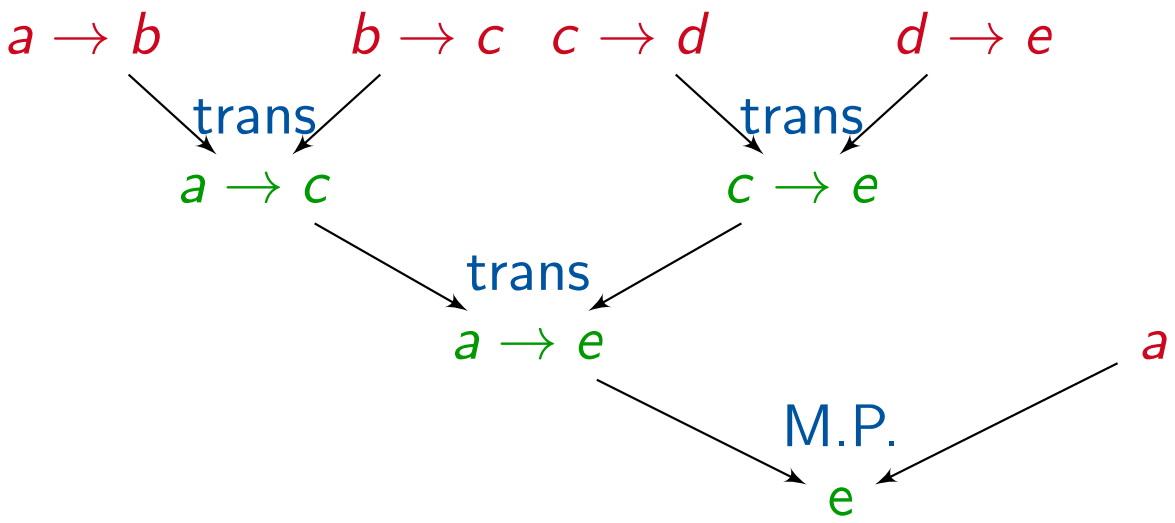
6.	$c \to e$	4, 5, *Trans*
7.	$a \to e$	3, 6, *Trans*
8.	a	*premise*
9.	e	7, 8, *M.P.*

Proof graph



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  With respect to the semantic definition of the logic. In the case of propositional logic truth tables give us this.

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- How to prove soundness and completeness?

Example: Hilbert axiom system (H)

- Let H be (M.P.) together with the following axiom schemes:

$$\frac{}{a \to (b \to a)} \quad (H1)$$

$$\frac{}{((a \to (b \to c)) \to ((a \to b) \to (a \to c)))} \quad (H2)$$

$$\frac{}{(\neg b \to \neg a) \to (a \to b)} \quad (H3)$$

- H is sound and complete for propositional logic.

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- Completeness: harder, but possible.

The resolution proof system

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- The resolution inference rule for CNF:

$$\frac{(l \vee l_1 \vee l_2 \vee \ldots \vee l_n) \quad (\neg l \vee l'_1 \vee \ldots \vee l'_m)}{(l_1 \vee \ldots \vee l_n \vee l'_1 \vee \ldots \vee l'_m)} \; \textit{Resolution}$$

- Example:

$$\frac{(a \vee b) \quad (\neg a \vee c)}{(b \vee c)}$$

- We first see some example proofs, before proving soundness and completeness.

Proof by resolution

- Let $\varphi = (a_1 \vee a_3) \wedge (\neg a_1 \vee a_2 \vee a_5) \wedge (\neg a_1 \vee a_4) \wedge (\neg a_1 \vee \neg a_4)$
- We want to prove $\varphi \rightarrow (a_3)$

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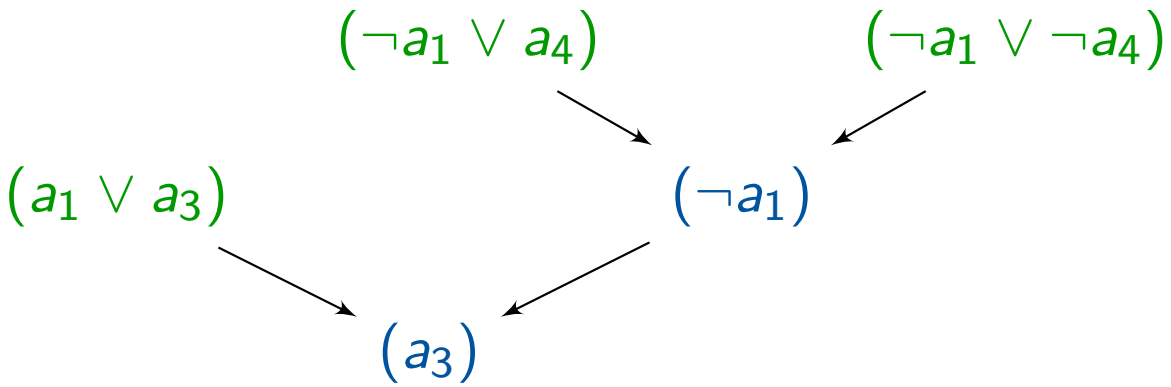
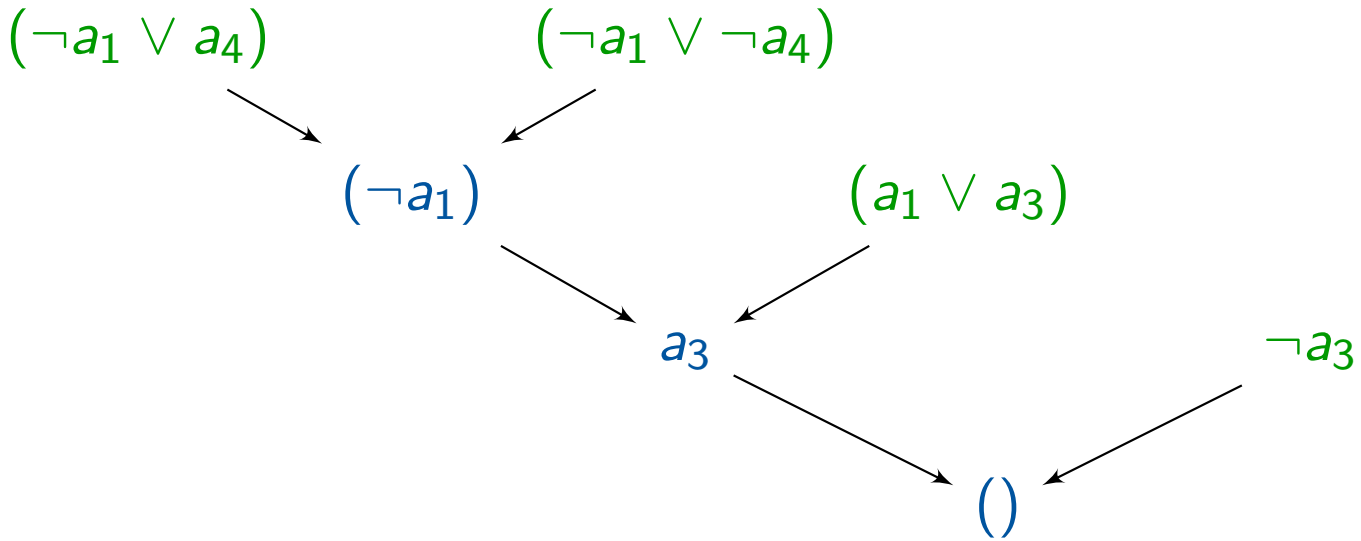


Resolution

- Resolution is a sound and complete proof system for CNF.
- If the input formula is unsatisfiable, there exists a proof of the empty clause.

Example

Let $\varphi = (a_1 \vee a_3) \wedge (\neg a_1 \vee a_2) \wedge (\neg a_1 \vee a_4) \wedge (\neg a_1 \vee \neg a_4) \wedge (\neg a_3)$.



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$$\models ((\varphi_1 \vee a) \wedge (\varphi_2 \vee \neg a)) \rightarrow (\varphi_1 \vee \varphi_2).$$

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- Completeness is a bit more involved.
  Basic idea: Use resolution for variable elimination.

$$\begin{gathered}(a \vee \varphi_1) \wedge \ldots \wedge (a \vee \varphi_n) \wedge \\ (\neg a \vee \psi_1) \wedge \ldots (\neg a \vee \psi_m) \wedge \\ R \\ \Leftrightarrow \\ (\varphi_1 \vee \psi_1) \wedge \ldots \wedge (\varphi_1 \vee \psi_m) \wedge \\ \ldots \\ (\varphi_n \vee \psi_1) \wedge \ldots (\varphi_n \vee \psi_m) \wedge \\ R\end{gathered}$$

  where φ_i $(i = 1, \ldots, n)$, ψ_j $(j = 1, \ldots, m)$, and R contains neither a nor $\neg a$.