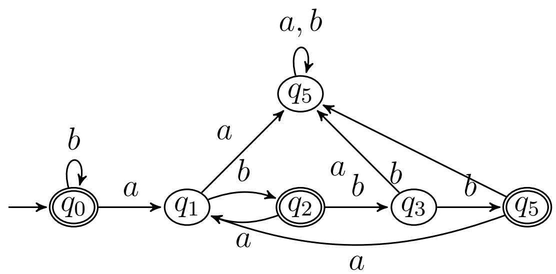
Exercises (Regular Languages)

1 Finite Automata

Exercise: Construct a DFA over $\Sigma := \{a, b\}$ that accepts the following language:

$$\{w \in \Sigma^* \mid \text{each } a \text{ followed by exactly 1 or 3 } b\text{'s}\}$$

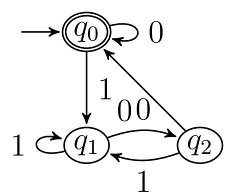
Solution:



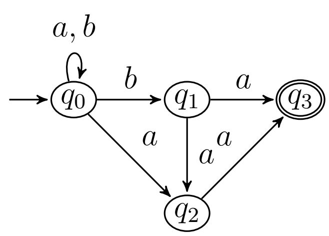
Exercise: Construct a DFA over $\Sigma := \{0, 1\}$ that accepts the following language:

$$\{w \in \Sigma^* \mid \text{decimal value of } w \text{ divisible by } 4\}$$

Solution: Equivalent: w of the form ε, 0, or $v00$ ($v \in \Sigma^*$). Thus:



Exercise: Let $\mathfrak{A}$ be the following NFA over $\Sigma := \{a, b\}$.

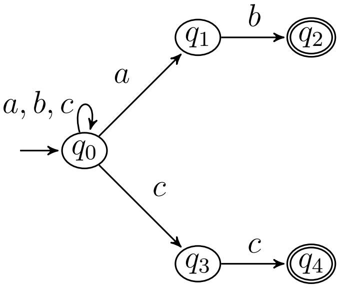


Determine the reachability sets $R_{\mathfrak{A}}(\varepsilon)$, $R_{\mathfrak{A}}(b)$, $R_{\mathfrak{A}}(ba)$, and $R_{\mathfrak{A}}(baa)$.

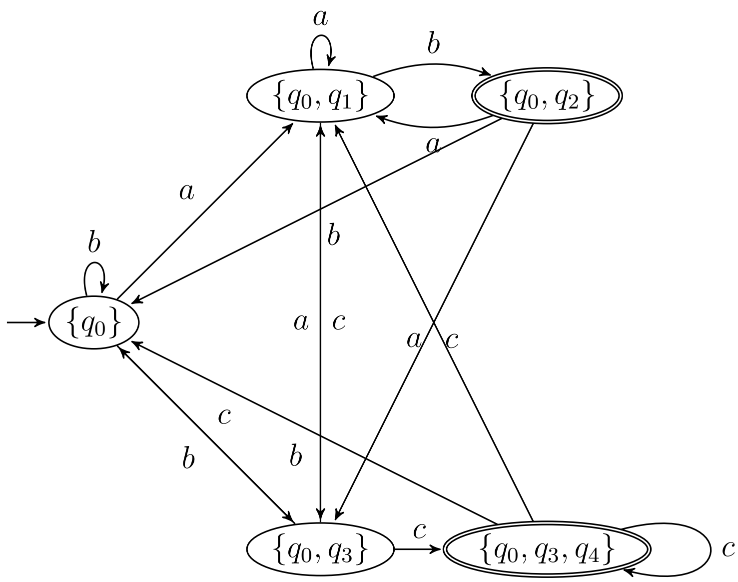
Solution:

$$\begin{aligned}
R_{\mathfrak{A}}(\varepsilon) &= \{q_0\} \\
R_{\mathfrak{A}}(b) &= \{q_0, q_1\} \\
R_{\mathfrak{A}}(ba) &= \{q_0, q_2, q_3\} \\
R_{\mathfrak{A}}(baa) &= \{q_0, q_2, q_3\}
\end{aligned}$$

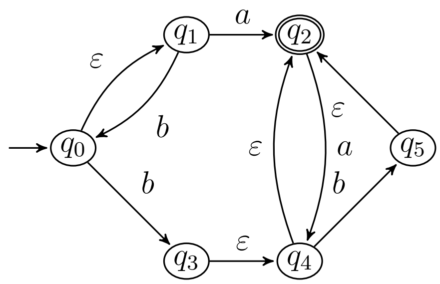
Exercise: Apply the powerset construction to transform the following NFA $\mathfrak{A}$ over $\Sigma := \{a, b, c\}$ into an equivalent DFA.



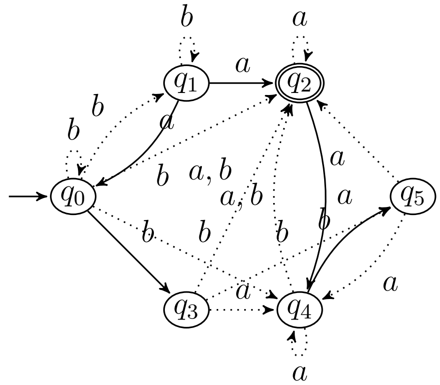
Solution:



Exercise: Eliminate all ε-transitions of the following ε-NFA $\mathfrak{A}$ over $\Sigma := \{a, b\}$ to obtain an equivalent NFA.



Solution:



2 Regular Expressions

Exercise: Give regular expressions that describe the following languages.

1. $L := \{w \in \{a,b\}^* \mid |w| \text{ divisible by } 3\}$
2. $L := \{w \in \{a,b,c\}^* \mid w \text{ does not contain } a, b, \text{ or } c\}$
3. $L := \{w \in \{a,b\}^* \mid \text{substring } ab \text{ occurs exactly twice in } w, \text{ but not at the end}\}$

Solution:

1. $((a+b)\cdot(a+b)\cdot(a+b))^*$
2. $(a+b)^* + (a+c)^* + (b+c)^*$
3. $b^*a^+b^+a^+b(b^+a^* + b^*a^+)$

Exercise: Show that regular languages are closed under the reversal operation.

Solution: Alternative proofs:

- For $L = L(\alpha)$ with regular expression α, $L^R = L(rev(\alpha))$ where
  - $rev(\emptyset) = \emptyset$
  - $rev(\varepsilon) = \varepsilon$
  - $rev(a) = a$
  - $rev(\alpha_1 + \alpha_2) = rev(\alpha_1) + rev(\alpha_2)$
  - $rev(\alpha_1 \cdot \alpha_2) = rev(\alpha_2) \cdot rev(\alpha_1)$
  - $rev(\alpha_1^*) = rev(\alpha_1)^*$
- For $L = L(\mathfrak{A})$ with DFA $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F\rangle$, ε-NFA $\mathfrak{A}^R = \langle Q', \Sigma, \Delta', q_0', F'\rangle$ with $\mathfrak{A}^R = L^R$ can be obtained as follows:
  - $Q' = Q \cup \{q_0'\}$
  - whenever $p \xrightarrow{a} q$ in Δ, $q \xrightarrow{a} p$ in Δ'

– additionally, for all $q \in F$ we let $q_0' \xrightarrow{\varepsilon} q$ in Δ'

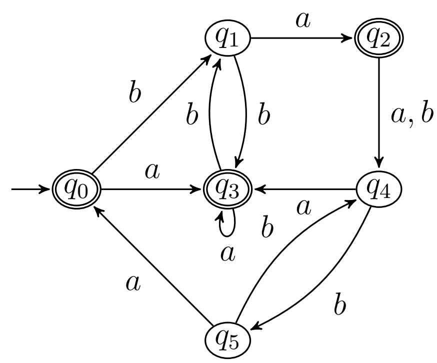
– $F' = \{q_0\}$

Remark: resulting automaton can be non-deterministic

Example: $\{w \in \{0,1\}^* \mid \text{decimal value of } w \text{ divisible by } 4\}$ (see previous exercise)

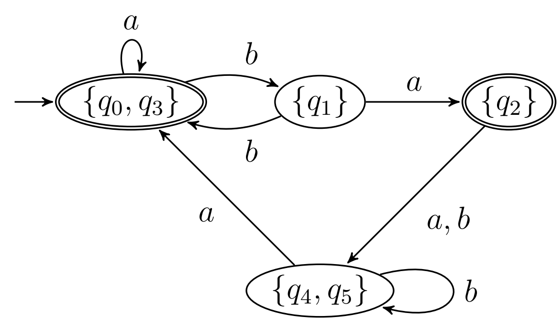
3 Minimisation of DFA

Exercise: Minimise the following DFA.



Solution:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	–	ε	a		ε	ε
q_1	–	–	ε	ε	b	aa
q_2	–	–	–	a	ε	ε
q_3	–	–	–	–	ε	ε
q_4	–	–	–	–	–	
q_5	–	–	–	–	–	–



4 Pumping Lemma

Exercise: Show that the following languages are *not* regular:

1. $L = \{a^p \mid p \text{ prime number}\}$
2. $L = \{a^{n^2} \mid n \geq 1\}$

Solution:

1. pumping argument: let $w = xy^{p-k}z \notin L$ for $p \geq n$ prime and $k = |y|$
2. If L regular, then there exists $n \geq 1$ such that every $w \in L$ with $|w| \geq n$ can be decomposed as $w = xyz$ where $y \neq \varepsilon$ and $xy^iz \in L$ for every $i \geq 0$.

   Given n, let $w = a^{n^2} \in L$ (such that $|w| = n^2 \geq n$). Hence w is decomposable as $w = xyz$ with $k := |y| > 0$.

Now for $i := 2n + k$ and $v := xy^i z$ we have:

$$\begin{aligned} |v| &= |xy^{2n+k}z| \\ &= |xz| + (2n+k)|y| \\ &= (n^2 - k) + (2n+k)k \\ &= n^2 + 2nk + k^2 - k \\ &= (n+k)^2 - k \end{aligned}$$

which cannot be a square number since the next lower one is

$$\begin{aligned} (n+k-1)^2 &= (n+k)^2 - 2(n+k) + 1 \\ &= |v| - 2n - k + 1 \\ &< |v| \end{aligned}$$