# Exercises (Context-Free Languages)

## 1 Context-Free Grammars

**Exercise:** Give context-free grammars that generate the following languages.

1. $L := \{a^k b^l c^{k+l} \mid k, l \in \mathbb{N}\}$
2. $L := \{a^k b^k c^l d^l \mid k, l \in \mathbb{N}\}$
3. $L := \{a^k b^l \mid k \geq 1, l > k\}$
4. $L := \{w \in \{a, b\}^* \mid |w| \text{ odd}, a \text{ in middle}\} (= \{uav \in \{a, b\}^* \mid |u| = |v|\})$

**Solution:**

1. $S \to aSc \mid B$
   $B \to bBc \mid \varepsilon$
2. $S \to AC$
   $A \to aAb \mid \varepsilon$
   $C \to cCd \mid \varepsilon$
3. $S \to AB$
   $A \to aAb \mid ab$
   $B \to bB \mid b$
4. $S \to aSa \mid aSb \mid bSa \mid bSb \mid a$ or shorter: $S \to ASA \mid a$
   $A \to a \mid b$

**Exercise:** Show that context-free languages are closed under the reversal operation.

**Solution:** Idea: reverse all right-hand sides of productions

## 2 Context-Free and Regular Languages

**Exercise:** Show that every regular expression can directly be translated into an equivalent context-free grammar.

**Solution:** Given regular expression $\alpha$ over $\Sigma$, we inductively construct CFG $G_\alpha = \langle N, \Sigma, P, S\rangle$ as follows:

- $\alpha = \emptyset$: $P = \emptyset$
- $\alpha = \varepsilon$: $P = \{S \to \varepsilon\}$
- $\alpha = a$: $P = \{S \to a\}$
- $\alpha = \alpha_1 + \alpha_2$: $P = \{S \to S_1 \mid S_2\} \cup P_1 \cup P_2$ (where $G_{\alpha_i} = \langle N_i, \Sigma, P_i, S_i\rangle$ for $i = 1, 2$)
- $\alpha = \alpha_1 \cdot \alpha_2$: $P = \{S \to S_1 S_2\} \cup P_1 \cup P_2$ (where $G_{\alpha_i}$ as above)
- $\alpha = \alpha_1^*$: $P = \{S \to S_1 S \mid \varepsilon\} \cup P_1$ (where $G_{\alpha_1}$ as above)