Exercises (Context-Free Languages)

1 Context-Free Grammars

1.1 **Exercise:** Give context-free grammars that generate the following languages.

1. $L := \{a^k b^l c^{k+l} \mid k, l \in \mathbb{N}_0\}$ (1) $S \to aSc \mid B \qquad B \to bBc \mid \varepsilon$
2. $L := \{a^k b^k c^l d^l \mid k, l \in \mathbb{N}_0\}$ $a^k \mid b^l \mid c^l \mid c^k$
3. $L := \{a^k b^l \mid k \geq 1, l > k\}$
4. $L := \{w \in \{a, b\}^* \mid |w| \text{ odd}, a \text{ in middle}\}(= \{uav \in \{a, b\}^* \mid |u| = |v|\})$ $u \mid a \mid v$

$S \to aSc \to a^2Sc^2 \to \dots$ (1) $a^k S c^k \to a^k b B c c^k \to a^k b^2 B c^2 c^k \to \dots \to a^k b^l B c^l c^k$

1.2 **Exercise:** Show that context-free languages are closed under the reversal operation.

2 Context-Free and Regular Languages

1.3 **Exercise:** Show that every regular expression can directly be translated into an equivalent context-free grammar.

1.1 Reminder:

$a^n = a^*$ $\quad S \to Aa$, $A \to Aa \mid \varepsilon$ $\quad$ (OR $S \to aA$, $A \to aA \mid \varepsilon$)

$a^n b^n$ $\quad S \to aAb \mid \varepsilon$, $A \to aAb \mid \varepsilon$

1. $P = \{S \to aSc \mid B,\ B \to bBc \mid \varepsilon\}$ 2. $P = \{S \to XY \mid \varepsilon,\ X \to aXb \mid \varepsilon,\ Y \to cYd \mid \varepsilon\}$

3. $P = \{S \to aSb \mid aBb,\ B \to Bb \mid b\}$ 4. $P = \{S \to ASA \mid a,\ A \to a \mid b\}$

$P' = \{S \to aSa \mid aSb \mid bSa \mid bSb \mid a\}$

5. $a^n b^{2n}$: $P = \{S \to aSbb \mid \varepsilon\}$

1.2 $w = a_1 \dots a_n \quad w^R = a_n \dots a_1$

L CF $\to$ exists CFG G s.t. $\mathcal{L}(G) = L$

$A \to \alpha_1 \dots \alpha_k \qquad A \to \alpha_k \dots \alpha_1$

$a^n b^n$: $S \to aSb \mid \varepsilon$

$S \to bSa \mid \varepsilon$: $b^n a^n$

1.3 regular expression: $\emptyset, \varepsilon, a \in \Sigma, \cdot, +, ^*$

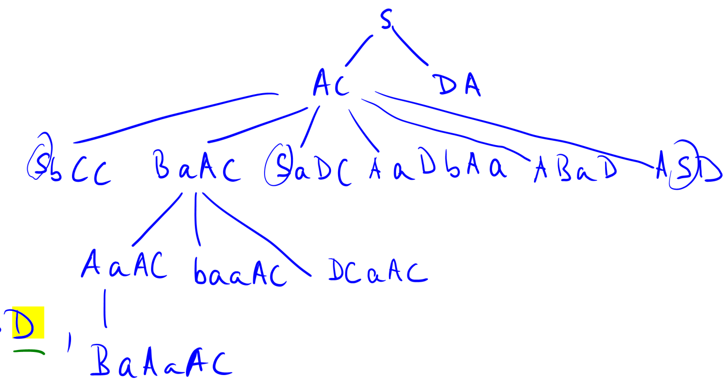
reg. exp. e

- $e = \emptyset \quad P = \emptyset$
- $e = \varepsilon \quad P = \{S \to \varepsilon\}$
- $e = a \quad P = \{S \to a\}$
- $e = e_1 + e_2 \quad P = \{S \to S_1 \mid S_2\} \cup P_1 \cup P_2$
- $e = e_1 \cdot e_2 \quad P = \{S \to S_1 S_2\} \cup P_1 \cup P_2$
- $e = e_1^* \quad P = \{S \to SS_1 \mid \varepsilon\} \cup P_1$

(1.4)

(i) Apply the marking algorithm to determine whether $L(G)=\emptyset$ for $G=(N,\Sigma,P,S)$,
$N=\{S,A,B,C,D\}$, $\Sigma=\{a,b\}$ and

$$
\begin{aligned}
P = \{\; S &\to AC \mid DA,\\
A &\to SbC \mid BaA \mid SaD,\\
B &\to A \mid ba \mid DC,\\
C &\to aDbAa \mid BaD \mid SD,\\
D &\to aB \mid CD \;\}
\end{aligned}
$$



(ii) List the elements from the set of terminating variables defined by

$\{X \in N \mid \exists w \in \Sigma^*.\ X \Rightarrow^* w\}$ $\{B, C, D\}$

(iii) If $L(G) \neq \emptyset$ then specify a word from $L(G)$.

1.5 Bring the following CFG in Chomsky normal form:

$S \to \varepsilon$
$X \to x$
$X \to YZ$

$S \to AB\boxed{a} \mid BAB \mid B$
$A \to SB \mid b \mid B\boxed{a}B$
$B \to ABS \mid A \mid \boxed{b}\,\boxed{a}$

$S_0 \to \boxed{S}$
$S \to ABN_a \mid BAB \mid \boxed{B}$
$A \to SB \mid b \mid BN_aB$
$B \to ABS \mid \boxed{A} \mid N_bN_a$
$N_a \to a$
$N_b \to b$

① Add new start symbol „S_0" and rule „$S_0 \to S$"

① Replace terminals „x" by auxiliary non-terminals „N_x" all rules, where the terminals do not appear „alone" and add rules „$N_x \to x$"

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② Remove chain rules $X \to Y$ and add for $Y \to \alpha$ also $X \to \alpha$.

$S \to ABN_a \mid BAB$
$A \to SB \mid b \mid BN_aB$
$B \to ABS \mid N_bN_a$
$N_a \to a$
$N_b \to b$

$S_0 \to ABN_a \mid BAB$ ($S_0 \to S$)
$B \to SB \mid b \mid BN_aB$ ($B \to A$)
$S \to ABS \mid N_bN_a$ ($S \to B$)

$S \to A\boxed{X_1} \mid B\boxed{X_2}$
$\boxed{X_1} \to BN_a$
$\boxed{X_2} \to AB$
$A \to SB \mid b \mid B\boxed{X_3}$
$\boxed{X_3} \to N_aB$
$B \to A\boxed{X_4} \mid N_bN_a$
$\boxed{X_4} \to BS$
$N_a \to a$
$N_b \to b$

$S_0 \to A\boxed{X_1} \mid B\boxed{X_2}$
$B \to SB \mid b \mid B\boxed{X_3}$
$S \to A\boxed{X_4} \mid N_bN_a$

③ Transform $X \to X_1 \cdots X_n$ with $n > 2$ to
$X \to X_1Y_1$
$Y_1 \to X_2Y_2$
$\vdots$
$Y_{n-2} \to X_{n-1}X_n$

(1.6) Consider the CF grammar G with $G = (\Sigma, N, P, S)$

$P = \{ S \to aS \mid Sb \mid a \mid b \}$ $\quad \Sigma = \{a, b\} \quad N = \{S\}$

i) Describe the set of all $w \in (N \cup \Sigma)^*$ that can be derived from S in G.

ii) Describe L(G).

iii) Is L(G) regular? Explain.

iv) Assume grammar G' with

$S \to a S'$

$S' \to Sb \mid a \mid b$

Describe L(G') and whether it is regular. Explain informally.

ii) $a^* (a|b) b^*$

i) $a^* (S|a|b) b^*$

iii) Yes (N.B. $a^n b^n$ not regular)

iv) $S \to aS' \to aSb$

aSb

aSb

$a(a|b)$

$a^n (aa \mid ab) b^n \quad n \geq 0$

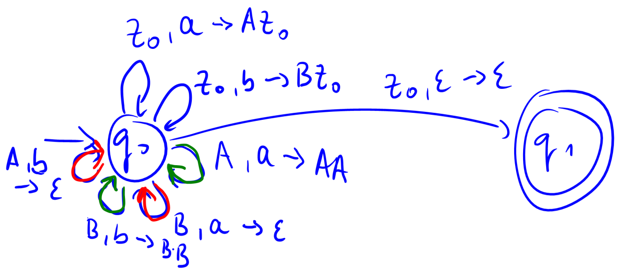
1.7 Assume the PDA $\mathcal{A} = (\underbrace{\{q_0, q_1\}}_{Q}, \underbrace{\{a,b\}}_{\Sigma}, \underbrace{\{Z_0, A, B\}}_{\Gamma}, q_0, Z_0, \Delta, \underbrace{\{q_1\}}_{F})$

with

$$\Delta = \{ (q_0, Z_0, a, q_0, AZ_0), (q_0, A, a, q_0, AA), (q_0, B, a, q_0, \varepsilon),$$
$$(q_0, Z_0, b, q_0, BZ_0), (q_0, B, b, q_0, BB), (q_0, A, b, q_0, \varepsilon),$$
$$(q_0, Z_0, \varepsilon, q_1, \varepsilon) \}$$

(i) Which of the words abba ✓, bbba ✗ and bbbaaa ✓ is accepted by $\mathcal{A}$?

(ii) Which language is accepted by $\mathcal{A}$?



$(q_0, Z_0, abba) \to (q_0, AZ_0, bba) \to (q_0, Z_0, ba)$
$\to (q_0, BZ_0, a) \to (q_0, Z_0, \varepsilon) \to (q_1, \varepsilon, \varepsilon)$

$(q_0, Z_0, bbba) \to (q_0, BZ_0, bba) \to (q_0, BBZ_0, ba) \to$
$(q_0, BBBZ_0, a) \to (q_0, BBZ_0, \varepsilon)$

$(q_0, Z_0, bbbaaa) \xrightarrow{*} (q_0, BBZ_0, aa) \to$
$(q_0, BZ_0, a) \to (q_0, Z_0, \varepsilon) \to (q_1, \varepsilon, \varepsilon)$

$\{ (a+b)^* \mid \#a = \#b \}$

1.8 CFG in Chomsky normal form:

$S \to AB \mid BC$
$A \to BA \mid a$
$B \to CC \mid b$
$C \to AB \mid a$

$\boxed{aaaaa} \in L(G)$?

	1	2	3	4	5
1	A,C	B	S,A,C	B	C,A,S
2	X	A,C	B	S,A,C	B
3	X	X	A,C	B	S,A,C
4	X	X	X	A,C	B
5	X	X	X	X	A,C

1-1 2-5
1-2 3-5
1-3 4-5
1-4 5-5

AB CB
BS BA BC
SB AB CB
BA BC

1 | 2 | 3 | 4 | 5

A,C | B

[1,3]
aaa
a|aa aa|a
[1,1] A,C [1,2] B
[2,3] B [3,3] A,C
AB BA
~~CB~~ BC

[1-4]
aaaa
a|aaa aa|aa
[1-1] [2-4] 1-2 3-4
A,C S,A,C B B
BB
AS
AA
AC
CS
CA
CC

aaa|a
1-3 4-4
S,A,C A,C
SA
SC