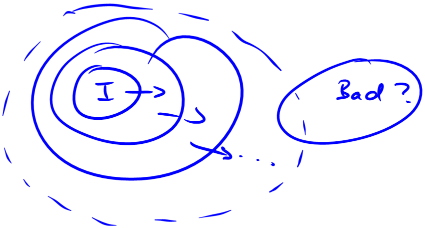
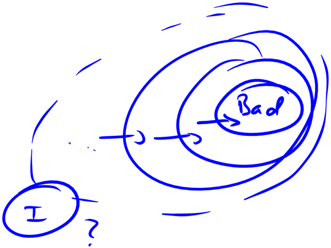
TA : Abstraction

init. RA : $\longrightarrow$ TA

Forward reach / Backward reach :





LHA I : Exact via logical encoding of state sets and transition relation

LHA II : $\dot{x} = Ax + Bu$ (logical encoding : dReach, iSAT-ODE)

Here: state set representation : geometrical objects

State set representation:

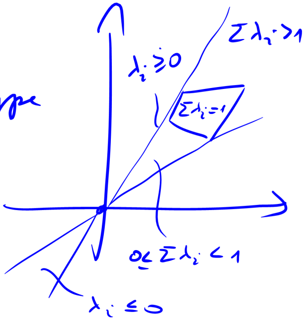
(1) Storage of state set
(2) Operations

} datatype

Here: Example polytopes

(1) H- or V-representation

(2) $\in$, $conv(\cdot \cup \cdot)$, $\cap$, lin. transform.
Minkowski sum $\oplus$, test for $\emptyset$



$V, \in$: $P = conv(V)$

$$= \{ \sum_i \lambda_i v_i \mid \lambda_i \in \mathbb{R}_{\geq 0}, \sum_i \lambda_i = 1 \}$$

$$Ax \leq b$$
$$\downarrow$$
$$(A, b)$$

$$(\exists \lambda_1, \dots, \lambda_n) \sum_i \lambda_i v_i = x \wedge \sum_i \lambda_i = 1 \wedge \lambda_1, \dots, \lambda_n \geq 0$$

Next $P \overset{?}{\to}$

Modeling and Analysis of Hybrid Systems

6. Convex polyhedra

Prof. Dr. Erika Ábrahám

Informatik 2 - LuFG Theory of Hybrid Systems
RWTH Aachen University

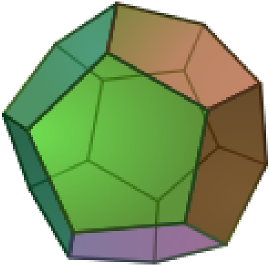
Szeged, Hungary, 27 September - 06 October 2017

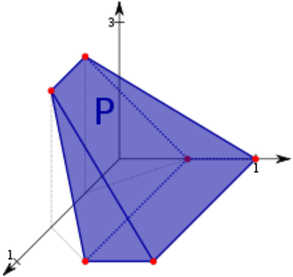
Contents

1 Convex polyhedra

2 Operations on convex polyhedra

(Convex) polyhedra and polytopes





(Convex) polyhedra and polytopes

Notation: In this lecture we use column-vector-notation, i.e., $\boldsymbol{x} \in \mathbb{R}^d$ is a column vector, whereas $\boldsymbol{x}^T$ is a row vector.

(Convex) polyhedra and polytopes

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A polyhedron in $\mathbb{R}^d$ is the solution set to a finite number $n \in \mathbb{N}_{\geq 0}$ of linear inequalities $\boldsymbol{a_i}^T \cdot \boldsymbol{x} \leq b_i$ (i.e., $a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i$), $i = 1, \ldots, n$, with real coefficients $\boldsymbol{a_i} \in \mathbb{R}^d$, real-valued variables $\boldsymbol{x} = (x_1, \ldots, x_d)^T$ and $b_i \in \mathbb{R}$. A bounded polyhedron is called a polytope.

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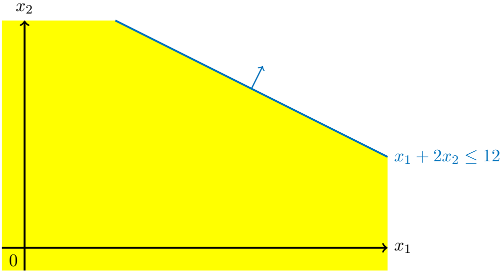

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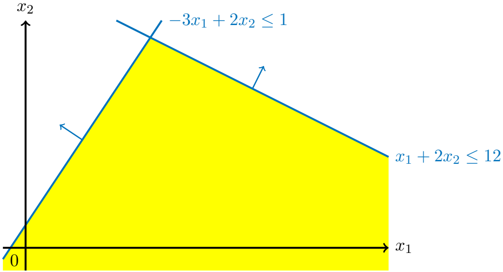

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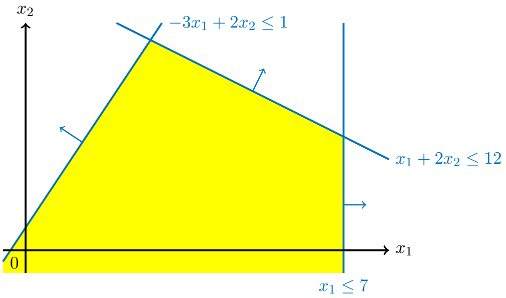
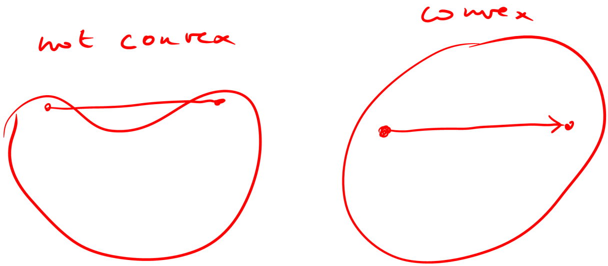


(Convex) polyhedra and polytopes

Definition (Reminder: convex sets)

A set $S \subseteq \mathbb{R}^d$ is convex if



(Convex) polyhedra and polytopes

Definition (Reminder: convex sets)

A set $S \subseteq \mathbb{R}^d$ is convex if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in S.\ \forall \lambda \in [0,1] \subseteq \mathbb{R}.\ \lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in S.$$

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Polyhedra are convex sets. Proof?

$$\bigwedge_{i=1}^{k} \vec{a}\,\vec{x} \le b$$

$$\vec{a}\,\vec{x}_1 \le b \qquad a\,\vec{x}_2 \le b$$

$$\vec{a}\left(\lambda\,\vec{x}_1 + (1-\lambda)\,\vec{x}_2\right) \le b$$

$$\lambda\,\vec{a}\,\vec{x}_1 + (1-\lambda)\,\underline{\vec{a}\,\vec{x}_2} \le b$$

$$\lambda\,\overline{\vec{a}\,\vec{x}_1} + (1-\lambda)\,\vec{a}\,\vec{x}_2 \le \lambda b + (1-\lambda) b = b$$

(Convex) polyhedra and polytopes

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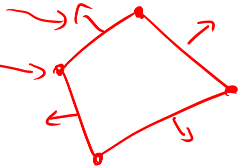
Polyhedra are convex sets. Proof?

Depending on the form of the representation we distinguish between

- $\mathcal{H}$-polytopes and
- $\mathcal{V}$-polytopes



$$\bigwedge_{i=1}^{k} \vec{a}_i \vec{x} \le b_i$$

$$\underline{A}\,\vec{x} \le \underline{\vec{b}}$$

$\mathcal{H}$-polytopes

Definition (Closed halfspace)

A d-dimensional closed halfspace is a set $\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^d \mid \boldsymbol{a}^T \cdot \boldsymbol{x} \leq b\}$ for some $\boldsymbol{a} \in \mathbb{R}^d$, called the normal of the halfspace, and some $b \in \mathbb{R}$.

$\mathcal{H}$-polytopes

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Definition ($\mathcal{H}$-polyhedron, $\mathcal{H}$-polytope)

A d-dimensional $\mathcal{H}$-polyhedron $P = \bigcap_{i=1}^{n} \mathcal{H}_i$ is the intersection of finitely many closed halfspaces. A bounded $\mathcal{H}$-polyhedron is called an $\mathcal{H}$-polytope.

The facets of a d-dimensional $\mathcal{H}$-polytope are $(d-1)$-dimensional $\mathcal{H}$-polytopes.

An $\mathcal{H}$-polytope

$$P \quad = \quad \bigcap_{i=1}^{n} \mathcal{H}_i \quad = \quad \bigcap_{i=1}^{n} \{\boldsymbol{x} \in \mathbb{R}^d \mid \boldsymbol{a_i}^T \cdot \boldsymbol{x} \leq b_i\}$$

can also be written in the form

$$P = \{\boldsymbol{x} \in \mathbb{R}^d \mid A\boldsymbol{x} \leq \boldsymbol{b}\}.$$

We call $(A, \boldsymbol{b})$ the $\mathcal{H}$-representation of the polytope.

$$2x + y \leq 3$$
$$x - 3y \leq 5$$

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} \leq \underbrace{\begin{pmatrix} 3 \\ 5 \end{pmatrix}}_{b}$$

$\mathcal{H}$-polytopes

An $\mathcal{H}$-polytope

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We call $(A, \boldsymbol{b})$ the $\mathcal{H}$-representation of the polytope.

- Each row of A is the normal vector to the ith facet of the polytope.

$\mathcal{H}$-polytopes

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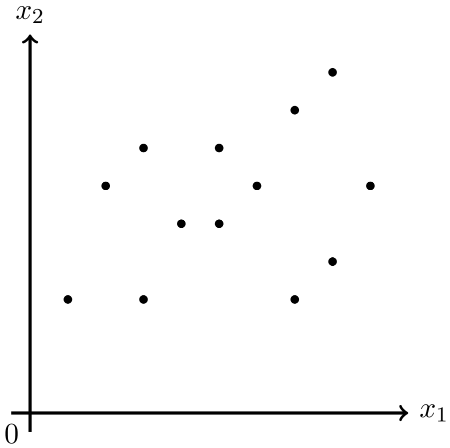
- Each row of A is the normal vector to the ith facet of the polytope.
- An $\mathcal{H}$-polytope P has a finite number of vertices $V(P)$.

Convex hull of a finite set of points



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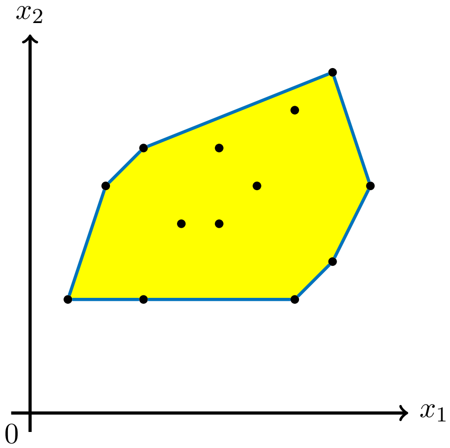

$\mathcal{V}$-polytopes

Definition (Convex hull)

Given a set $V \subseteq \mathbb{R}^d$, the convex hull $conv(V)$ of V is the smallest convex set that contains V.

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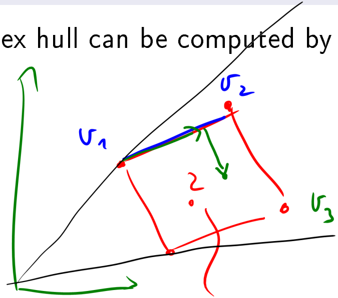

linear combination of vertices

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$$conv(V) = \{\boldsymbol{x} \in \mathbb{R}^d \mid \exists \lambda_1, \ldots, \lambda_n \in [0,1] \subseteq \mathbb{R}.\ \sum_{i=1}^{n} \lambda_i = 1 \wedge \sum_{i=1}^{n} \lambda_i \boldsymbol{v_i} = \boldsymbol{x}\}.$$

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Note that all $\mathcal{V}$-polytopes are bounded.

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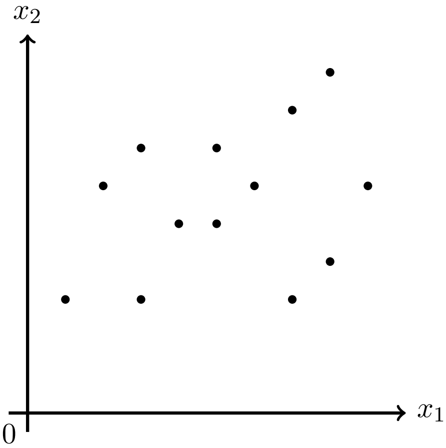
Note that all $\mathcal{V}$-polytopes are bounded.
Unbounded polyhedra can be represented by extending convex hulls with conical hulls.

Conical hull of a finite set of points



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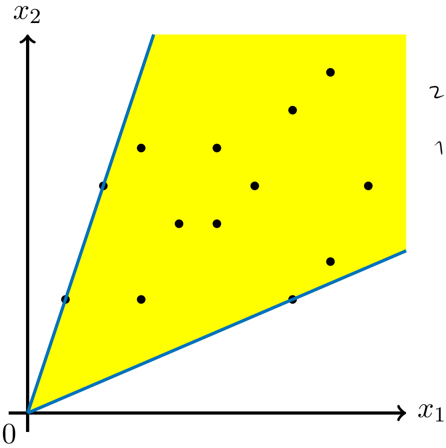

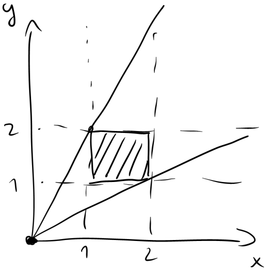

$\dot{x} = [1,2]$

$\dot{y} = [1,2]$

Conical hull of a finite set of points

Definition (Conical hull)

If $U = \{\boldsymbol{u_1}, \ldots, \boldsymbol{u_n}\}$ is a finite set of points in $\mathbb{R}^d$, the conical hull of U is defined by

$$cone(U) = \{\boldsymbol{x} \in \mathbb{R}^d \mid \exists \lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}.\ \boldsymbol{x} = \sum_{i=1}^{n} \lambda_i \boldsymbol{u_i}\}. \quad (1)$$

Conical hull of a finite set of points

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Each polyhedra $P \subseteq \mathbb{R}^d$ can be represented by two finite sets $V, U \subseteq \mathbb{R}^d$ such that

$$P = conv(V) \oplus cone(U) = \{\boldsymbol{v} + \boldsymbol{u} \mid \boldsymbol{v} \in conv(V),\ \boldsymbol{u} \in cone(U)\}\ ,$$

where $\oplus$ is called the Minkowski sum. $A \oplus B = \{a + b \mid a \in A, b \in B\}$

If U is empty then P is bounded (e.g., a polytope).

In the following we consider, for simplicity, only bounded $\mathcal{V}$-polytopes.

Motzkin's theorem

- For each $\mathcal{H}$-polytope, the convex hull of its vertices defines the same set in the form of a $\mathcal{V}$-polytope, and vice versa,
- each set defined as a $\mathcal{V}$-polytope can be also given as an $\mathcal{H}$-polytope by computing the halfspaces defined by its facets.

The translations between the $\mathcal{H}$- and the $\mathcal{V}$-representations of polytopes can be exponential in the state space dimension d.

Contents

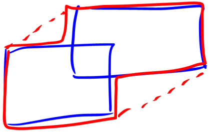
1 Convex polyhedra

2 Operations on convex polyhedra

Operations

If we represent reachable sets of hybrid automata by polytopes, we need some operations like

- membership computation ($\boldsymbol{x} \in P$)
- linear transformation ($A \cdot P$)
- Minkowski sum ($P_1 \oplus P_2$)
- intersection ($P_1 \cap P_2$)
- (convex hull of the) union of two polytopes ($conv(P_1 \cup P_2)$)
- test for emptiness ($P = \emptyset$?)



Membership $\boldsymbol{p} \in P$

Membership for $\boldsymbol{p} \in \mathbb{R}^d$:

$\mathcal{V}$: set of vertices $V = \{\vec{v}_1, \ldots, \vec{v}_n\}$:
point $\vec{p}$

$$\boxed{\begin{array}{l} \exists \lambda_1, \ldots, \lambda_n . \\ \vec{p} = \sum_{i=1}^{n} \lambda_i \vec{v}_i \\ \sum_{i=1}^{n} \lambda_i = 1 \end{array}} \rightarrow$$

conjunctive
linear
real $\rightarrow$ polyn.
arithm. compl.

$\mathcal{H}$: $(A, \vec{b})$:
point $\vec{p}$

$$\boxed{A \vec{p} \leq \vec{b}} \rightarrow$$

only constants
no variables
$\downarrow$
linear complexity

Membership $\boldsymbol{p} \in P$

Membership for $\boldsymbol{p} \in \mathbb{R}^d$:

- $\mathcal{H}$-polytope defined by $A\boldsymbol{x} \leq \boldsymbol{z}$:

Membership $\boldsymbol{p} \in P$

Membership for $\boldsymbol{p} \in \mathbb{R}^d$:

- $\mathcal{H}$-polytope defined by $A\boldsymbol{x} \leq \boldsymbol{z}$:
  just substitute $\boldsymbol{p}$ for $\boldsymbol{x}$ to check if the inequation holds.

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$$\exists \lambda_1, \ldots, \lambda_n \in [0,1] \subseteq \mathbb{R}. \sum_{i=1}^{n} \lambda_i = 1 \wedge \sum_{i=1}^{n} \lambda_i \boldsymbol{v_i} = \boldsymbol{x} \ .$$

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Alternatively:

Membership $\boldsymbol{p} \in P$

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Alternatively: convert the $\mathcal{V}$-polytope into an $\mathcal{H}$-polytope by computing its facets. → possibly exponential compl.

Intersection $P_1 \cap P_2$

Intersection for two polytopes P_1 and P_2:

- $\mathcal{H}$-polytopes defined by $A_1\boldsymbol{x} \leq \boldsymbol{b_1}$ and $A_2\boldsymbol{x} \leq \boldsymbol{b_2}$:

$$\left(\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right)$$

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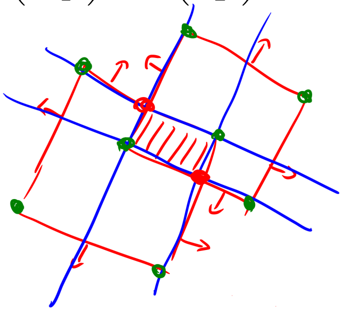
- $\mathcal{V}$-polytopes defined by V_1 and V_2:



Intersection $P_1 \cap P_2$

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- $\mathcal{V}$-polytopes defined by V_1 and V_2:
  Convert P_1 and P_2 to $\mathcal{H}$-polytopes and convert the result back to a $\mathcal{V}$-polytope.

Union $P_1 \cup P_2$

Note that the union of two convex polytopes is in general not a convex polytope.

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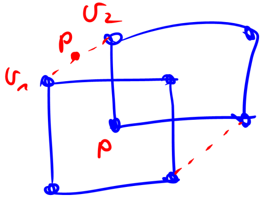
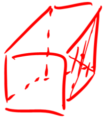


$$\exists \lambda_1,\dots,\lambda_k \cdot p = \sum_{i=1}^{k} \lambda_i v_i$$

$$\sum_{i=1}^{k} \lambda_i \leq 1$$



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  $\mathcal{V}$-representation $V_1 \cup V_2$.
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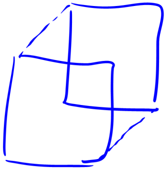

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- $\mathcal{V}$-polytopes defined by V_1 and V_2:
  $\mathcal{V}$-representation $V_1 \cup V_2$.
- $\mathcal{H}$-polytopes defined by $A_1\boldsymbol{x} \leq \boldsymbol{b_1}$ and $A_2\boldsymbol{x} \leq \boldsymbol{b_2}$:
  convert to $\mathcal{V}$-polytopes and compute back the result.

$P = \emptyset$?

V: $V = \emptyset$

- yes: $P = \emptyset$
- no: $P \neq \emptyset$

<u>constant</u>

H: $(A, \vec{b})$ $\quad \exists \vec{x}.\ A\vec{x} \le \vec{b}$

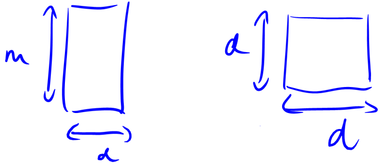
<u>polynomial</u>

$A \cdot P$ linear transformation

V: $\{A \cdot v \mid v \in V\}$

H: $(A', \vec{b}) \to V \to A \cdot V \to H_{A\cdot V}$

$$\underset{d}{x} = \underset{d \times d}{A}\ \underset{d}{x}$$



$\{x \mid A'x \le b\}$

$\{Ax \mid A'x \le b\}$

$\{\cancel{x}\, y \mid A'(A^{-1}y) \le b\}$

$y = Ax$

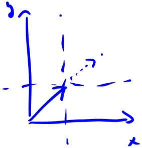
$A^{-1}y = x$

does not always exist

might be hard to compute(?)

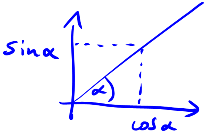
Linear transformation:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



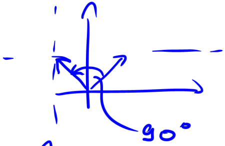
$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$



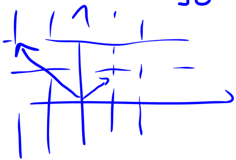
$$R_{90} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$R_{90} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



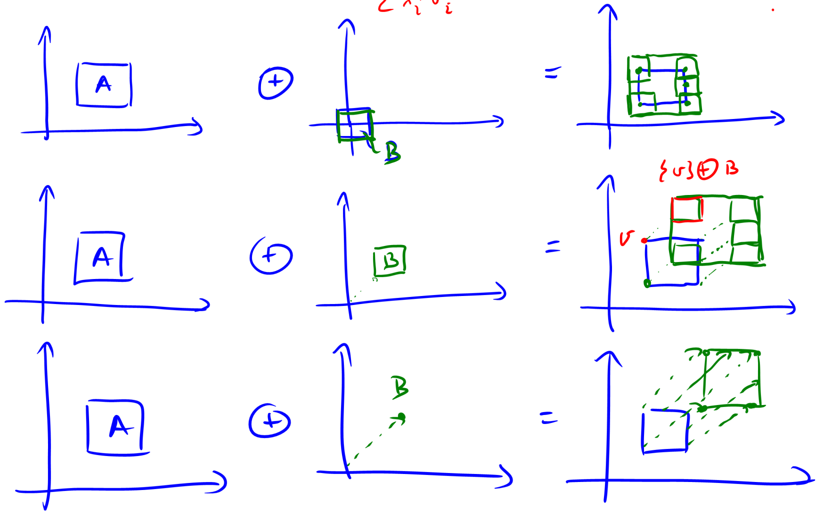
$$2 \cdot R_{90} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$$(2 \cdot R_{90})\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$



$$A \oplus B = \{a + b \mid a \in A,\ b \in B\}$$

$$\sum \lambda_i v_i^a$$



Lemma Assume $V_A = \{v_{A,1}, \dots, v_{A,k}\}$, $V_B = \{v_{B,1}, \dots, v_{B,\ell}\}$.

Then

$$\operatorname{conv}(V_A) \oplus \operatorname{conv}(V_B) = \operatorname{conv}(V_A \oplus V_B)$$

Proof

„$\supseteq$" Let $x \in \operatorname{conv}(V_A \oplus V_B)$. Then there exist some $\sum_{i=1}^{k} \sum_{j=1}^{\ell} \lambda_{i,j} = 1$ we have

$$\begin{aligned} x &= \sum_{i=1}^{k} \sum_{j=1}^{\ell} \lambda_{i,j} (v_{A,i} + v_{B,j}) \\ &= \Big(\sum_{i=1}^{k} \underbrace{\Big(\sum_{j=1}^{\ell} \lambda_{i,j} \Big)}_{\alpha_i} v_{A,i} \Big) + \Big(\sum_{j=1}^{\ell} \underbrace{\Big(\sum_{i=1}^{k} \lambda_{i,j} \Big)}_{\beta_j} v_{B,j} \Big) \\ &= \Big(\sum_{i=1}^{k} \alpha_i v_{A,i} \Big) + \Big(\sum_{j=1}^{\ell} \beta_j v_{B,j} \Big) \in \operatorname{conv}(V_A) \oplus \operatorname{conv}(V_B) \end{aligned}$$

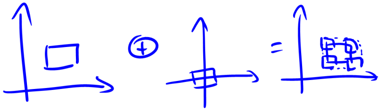
(Note: $\sum_{i=1}^{k} \alpha_i = \sum_{j=1}^{\ell} \beta_j = 1$.)

„$\subseteq$" Let $x \in \operatorname{conv}(V_A) \oplus \operatorname{conv}(V_B)$. Then there exist some $\sum_{i=1}^{k} \alpha_i = 1$ and $\sum_{j=1}^{\ell} \beta_j = 1$ with

$$\begin{aligned} x &= \Big(\sum_i \alpha_i v_{A,i}\Big) + \Big(\sum_j \beta_j v_{B,j}\Big) = \Big(\sum_i \underbrace{\Big(\sum_j \beta_j\Big)}_{=1} \alpha_i v_{A,i}\Big) + \Big(\sum_j \underbrace{\Big(\sum_i \alpha_i\Big)}_{=1} \beta_j v_{B,j}\Big) \\ &= \sum_i \sum_j \alpha_i \beta_j (v_{A,i} + v_{B,j}) \in \operatorname{conv}(V_A \oplus V_B). \quad \Big(\text{Note: } \sum_i \sum_j \alpha_i \beta_j = 1\Big) \end{aligned}$$

$\square$

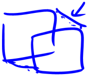
Operation complexity overview



"easy": doable in polynomial complexity
"hard": no polynomial algorithm is known

	$x \in P$	$A \cdot P$	$P_1 \oplus P_2$	$P_1 \cap P_2$	conv$(P_1 \cup P_2)$	$=\emptyset$?
$\mathcal{V}$-polytope	easy	easy	–	–	–	e
$\mathcal{H}$-polytope	easy	hard	–	–	–	e
$\mathcal{V}$-polytope and $\mathcal{V}$-polytope	–	–	easy	hard	easy	–
$\mathcal{H}$-polytope and $\mathcal{H}$-polytope	–	–	hard	easy	hard	–
$\mathcal{V}$-polytope and $\mathcal{H}$-polytope	–	–	hard	hard	hard	–

It is in general also hard to translate a $\mathcal{V}$-polytope to an $\mathcal{H}$-polytope or vice versa.



Why linear transformation:

initial state set P , $\dot{x} = Ax$

at time $\delta > 0$ system reaches states

$$\boxed{e^{A\delta} \cdot P}$$

matrix exponential