Modeling and Analysis of Hybrid Systems

6. Convex polyhedra

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Informatik 2 - LuFG Theory of Hybrid Systems
RWTH Aachen University

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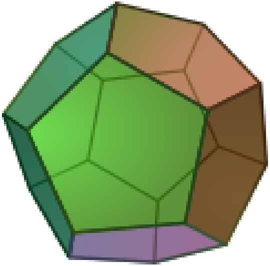
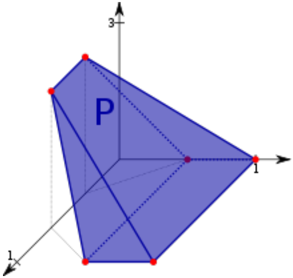
Contents

(Convex) polyhedra and polytopes





(Convex) polyhedra and polytopes

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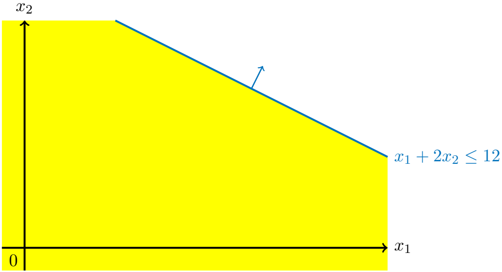

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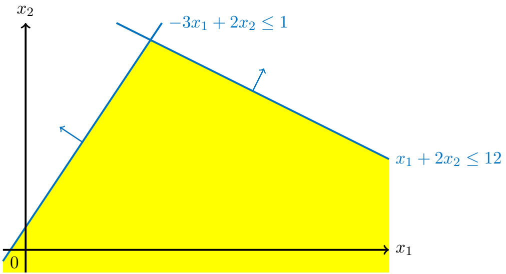

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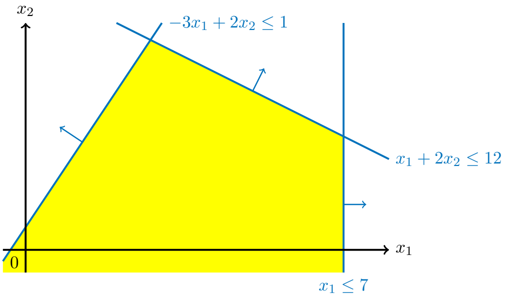

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Depending on the form of the representation we distinguish between

- $\mathcal{H}$-polytopes and
- $\mathcal{V}$-polytopes

$\mathcal{H}$-polytopes

Definition (Closed halfspace)

A d-dimensional closed halfspace is a set $\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^d \mid \boldsymbol{a}^T \cdot \boldsymbol{x} \leq b\}$ for some $\boldsymbol{a} \in \mathbb{R}^d$, called the normal of the halfspace, and some $b \in \mathbb{R}$.

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Definition ($\mathcal{H}$-polyhedron, $\mathcal{H}$-polytope)

A d-dimensional $\mathcal{H}$-polyhedron $P = \bigcap_{i=1}^{n} \mathcal{H}_i$ is the intersection of finitely many closed halfspaces. A bounded $\mathcal{H}$-polyhedron is called an $\mathcal{H}$-polytope.

The facets of a d-dimensional $\mathcal{H}$-polytope are $d-1$-dimensional $\mathcal{H}$-polytopes.

$\mathcal{H}$-polytopes

An $\mathcal{H}$-polytope

$$P = \bigcap_{i=1}^{n} \mathcal{H}_i = \bigcap_{i=1}^{n} \{\boldsymbol{x} \in \mathbb{R}^d \mid \boldsymbol{a_i}^T \cdot \boldsymbol{x} \leq b_i\}$$

can also be written in the form

$$P = \{\boldsymbol{x} \in \mathbb{R}^d \mid A\boldsymbol{x} \leq \boldsymbol{b}\}.$$

We call $(A, \boldsymbol{b})$ the $\mathcal{H}$-representation of the polytope.

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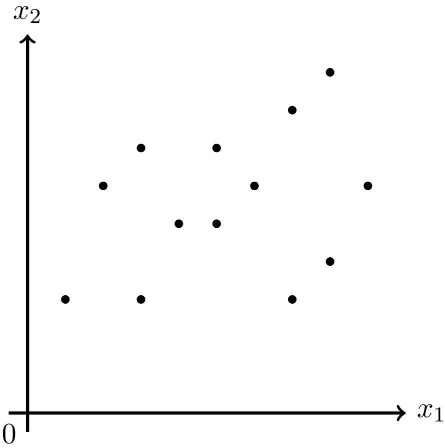
- An $\mathcal{H}$-polytope P has a finite number of vertices $V(P)$.

Convex hull of a finite set of points



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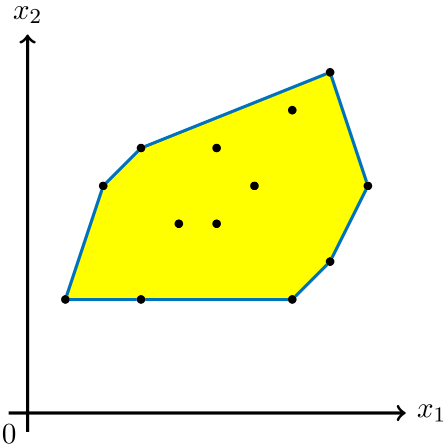

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Unbounded polyhedra can be represented by extending convex hulls with conical hulls.

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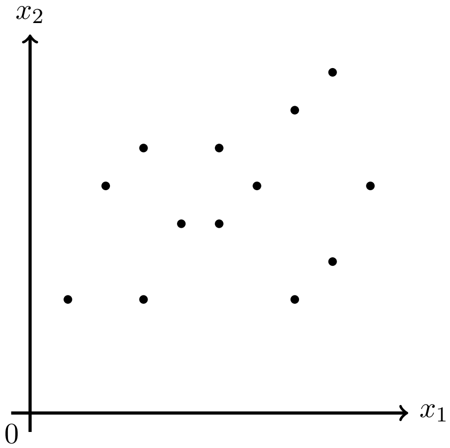

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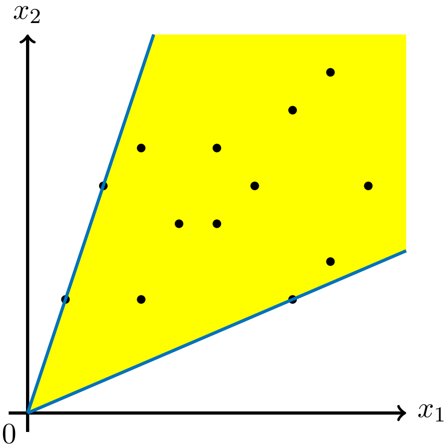

Conical hull of a finite set of points

Definition (Conical hull)

If $U = \{\boldsymbol{u_1}, \ldots, \boldsymbol{u_n}\}$ is a finite set of points in $\mathbb{R}^d$, the conical hull of U is defined by

$$cone(U) = \{\boldsymbol{x} \in \mathbb{R}^d \mid \exists \lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}.\ \boldsymbol{x} = \sum_{i=1}^{n} \lambda_i \boldsymbol{u_i}\}. \qquad (1)$$

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Each polyhedra $P \subseteq \mathbb{R}^d$ can be represented by two finite sets $V, U \subseteq \mathbb{R}^d$ such that

$$P = conv(V) \oplus cone(U) = \{\boldsymbol{v} + \boldsymbol{u} \mid \boldsymbol{v} \in conv(V),\ \boldsymbol{u} \in cone(U)\}\ ,$$

where $\oplus$ is called the Minkowski sum.
If U is empty then P is bounded (e.g., a polytope).
In the following we consider, for simplicity, only bounded $\mathcal{V}$-polytopes.

Motzkin's theorem

- For each $\mathcal{H}$-polytope, the convex hull of its vertices defines the same set in the form of a $\mathcal{V}$-polytope, and vice versa,
- each set defined as a $\mathcal{V}$-polytope can be also given as an $\mathcal{H}$-polytope by computing the halfspaces defined by its facets.

The translations between the $\mathcal{H}$- and the $\mathcal{V}$-representations of polytopes can be exponential in the state space dimension d.

Contents

Operations

If we represent reachable sets of hybrid automata by polytopes, we need some operations like

- membership computation ($\boldsymbol{x} \in P$)
- linear transformation ($A \cdot P$)
- Minkowski sum ($P_1 \oplus P_2$)
- intersection ($P_1 \cap P_2$)
- (convex hull of the) union of two polytopes (*conv*$(P_1 \cup P_2)$)
- test for emptiness ($P = \emptyset$?)

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Alternatively: convert the $\mathcal{V}$-polytope into an $\mathcal{H}$-polytope by computing its facets.

Intersection $P_1 \cap P_2$

Intersection for two polytopes P_1 and P_2:

- $\mathcal{H}$-polytopes defined by $A_1\boldsymbol{x} \leq \boldsymbol{b_1}$ and $A_2\boldsymbol{x} \leq \boldsymbol{b_2}$:

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  the resulting $\mathcal{H}$-polytope is defined by $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \boldsymbol{x} \leq \begin{pmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \end{pmatrix}$.
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- $\mathcal{V}$-polytopes defined by V_1 and V_2:
  Convert P_1 and P_2 to $\mathcal{H}$-polytopes and convert the result back to a $\mathcal{V}$-polytope.

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  $\mathcal{V}$-representation $V_1 \cup V_2$.
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  convert to $\mathcal{V}$-polytopes and compute back the result.

Operation complexity overview

“easy”: doable in polynomial complexity
“hard”: no polynomial algorithm is known

	$x \in P$	$A \cdot P$	$P_1 \oplus P_2$	$P_1 \cap P_2$	$P_1 \cup P_2$
$\mathcal{V}$-polytope	easy	easy	—	—	—
$\mathcal{H}$-polytope	easy	hard	—	—	—
$\mathcal{V}$-polytope and $\mathcal{V}$-polytope	—	—	easy	hard	easy
$\mathcal{H}$-polytope and $\mathcal{H}$-polytope	—	—	hard	easy	hard
$\mathcal{V}$-polytope and $\mathcal{H}$-polytope	—	—	hard	hard	hard

It is in general also hard to translate a $\mathcal{V}$-polytope to an $\mathcal{H}$-polytope or vice versa.