Modeling and Analysis of Hybrid Systems

Series 6

Exercise 1

A n-dimensional box $\mathcal{B}$ can be defined as a vector of intervals:

$$\mathcal{B} = I_0 \times \ldots \times I_n, I_i \in \mathbb{I}$$

where each interval is defined as $I_i = [l_i, u_i], l_i \leq u_i, u_i \in \mathbb{R} \cup \{\infty\}, l_i \in \mathbb{R} \cup \{-\infty\}$ such that $x \in I \Leftrightarrow l_i \leq x \leq u_i$.

Please give a suitable definition for the operations

a) union,

b) intersection,

c) test for membership.

Solution:

We denote I_{a_i} as the i-th interval of a given box $\mathcal{A}$ and its bounds as a_{l_i} and a_{u_i}.

a) The union of two boxes $\mathcal{A}$ and $\mathcal{B}$ is defined as the smallest box containing the convex hull of both boxes, as boxes (and convex objects in general) are not closed under the operation *union*. As the dimensions are independent, the operation can be performed component-wise:

$$\begin{aligned}\mathcal{A} \cup \mathcal{B} &= conv(I_{a_0}, I_{b_0}) \times \ldots \times conv(I_{a_n}, I_{b_n}) \\ &= [\min(a_{l_0}, b_{l_0}), \max(a_{u_0}, b_{u_0})] \times \ldots \times [\min(a_{l_n}, b_{l_n}), \max(a_{u_n}, b_{u_n})].\end{aligned}$$

b) As the union, the intersection of two boxes $\mathcal{A}$ and $\mathcal{B}$ can also be defined component-wise:

$$\mathcal{A} \cap \mathcal{B} = (I_{a_0} \cap I_{b_0}) \times \ldots \times (I_{a_n} \cap I_{b_n})$$

where the intersection of two intervals I_a, I_b is defined as

$$I_a \cap I_b = \begin{cases} [l_b, u_a] & \text{for } u_b \leq u_a \leq l_b, \\ [l_a, u_b] & \text{for } u_a \geq u_b \geq l_a, \\ [l_a, u_a] & \text{for } l_b \leq l_a \leq u_a \leq u_b, \\ [l_b, u_b] & \text{for } l_a \leq l_b \leq u_b \leq u_a, \\ \emptyset & \text{else.} \end{cases}$$

c) The test for membership for a point $p \in \mathbb{R}^n$ and a box $\mathcal{A}$ can as well be performed component-wise such that:

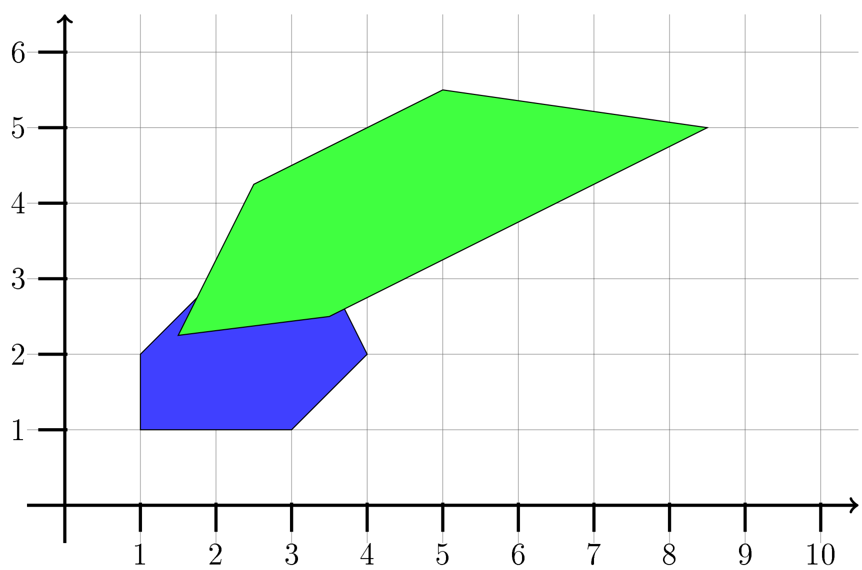
$$\begin{aligned} p \in \mathcal{A} &\Leftrightarrow p_i \in I_{a_i}, 0 \le i \le n \\ &\Leftrightarrow a_{l_i} \le p_i \le a_{u_i}, 0 \le i \le n \end{aligned}$$

Exercise 2

In the lecture we presented bounded $\mathcal{V}$-polytopes. Please perform the following operations on the given set representations. For each task, please give the vertices of the resulting polytope and a sketch in the provided canvas.

a) Given a $\mathcal{V}$-polytope $\mathcal{P} = cHull\left(\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}\right)$ as shown below. Please compute the resulting polytope $\mathcal{P}'$ after a linear transformation $Ax + b$ with

$$A = \begin{pmatrix}0.5 & 2\\1 & 0.25\end{pmatrix}, b = \begin{pmatrix}-1\\1\end{pmatrix}.$$



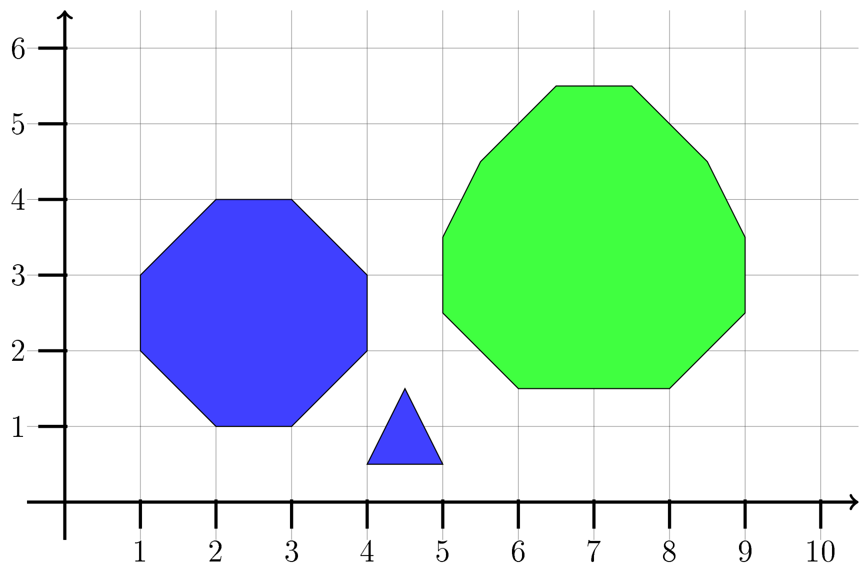
Solution: The resulting polytope is

$$\mathcal{P}' = cHull\left(\left\{\begin{pmatrix}1.5\\2.25\end{pmatrix}, \begin{pmatrix}2.5\\4.25\end{pmatrix}, \begin{pmatrix}5\\5.5\end{pmatrix}, \begin{pmatrix}8.5\\5\end{pmatrix}, \begin{pmatrix}3.5\\2.5\end{pmatrix}\right\}\right).$$

b) Given the $\mathcal{V}$-polytopes

$$\mathcal{A} = cHull\left(\left\{\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}4\\3\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}\right) \text{ and}$$
$$\mathcal{B} = cHull\left(\left\{\begin{pmatrix}4\\0.5\end{pmatrix}, \begin{pmatrix}5\\0.5\end{pmatrix}, \begin{pmatrix}4.5\\1.5\end{pmatrix}\right\}\right)$$

the as shown below. Please compute the resulting polytope $\mathcal{P}'$ after a Minkowski sum of both polytopes, i.e. compute $\mathcal{P}' = \mathcal{A} \oplus \mathcal{B}$.



Solution: The resulting polytope is

$$\mathcal{P}' = cHull\left(\left\{\begin{pmatrix}6\\1.5\end{pmatrix}, \begin{pmatrix}8\\1.5\end{pmatrix}, \begin{pmatrix}9\\2.5\end{pmatrix}, \begin{pmatrix}9\\3.5\end{pmatrix}, \begin{pmatrix}8.5\\4.5\end{pmatrix}, \begin{pmatrix}7.5\\5.5\end{pmatrix}, \begin{pmatrix}6.5\\5.5\end{pmatrix}, \begin{pmatrix}5.5\\4.5\end{pmatrix}, \begin{pmatrix}5\\3.5\end{pmatrix}, \begin{pmatrix}5\\2.5\end{pmatrix}\right\}\right).$$

Exercise 3

An $\mathcal{H}$-polytope is represented by a set of linear inequalities (constraints). Could you design an (abstract) algorithm to remove the redundant inequalities for an $\mathcal{H}$-polytope? Similarly, could you give an (abstract) algorithm to remove the redundant vertices for a $\mathcal{V}$-polytope.

Solution:

(1) We use the following method to check whether a constraint $L_i : c_i^T x \leq z_i$ is redundant in $S = \{L_1, \ldots, L_n\}$.
We compute the following linear program

$$z = \max(c_i^T x) \quad \text{subject to} \quad c_k^T x \leq z_k \text{ for } 1 \leq k \leq n \wedge k \neq i$$

The constraint L_i is redundant in S if and only if $z \leq z_i$.
Assume the $\mathcal{H}$-polytope P is represented by the constraint set S_P. We use the introduced method to check every constraint in S_P and remove the redundant ones.

(2) The work of removing redundant vertices is similar. We use the following method to check whether a vertex v_i is redundant in the set $V = v_1, \ldots, v_n$.
We check the following linear feasibility problem

$$\begin{aligned} \text{find} \quad & \text{all } \lambda_j \geq 0 \text{ where } 1 \leq j \leq n \text{ and } j \neq i \\ \text{subject to} \quad & v_i = \sum_{1 \leq j \leq n, j \neq i} \lambda_j v_j \ \wedge \sum_{1 \leq j \leq n, j \neq i} \lambda_j = 1 \end{aligned}$$

The vertex v_i is redundant in V if and only if the above problem is feasible.
Assume the $\mathcal{V}$-polytope P is represented by the vertex set V_P. We use the introduced method to check every vertex in V_P and remove the redundant ones.

Exercise 4

a) A 2-dimensional polyhedron P is defined by the following linear inequalities.

$$\begin{cases} -x & & & \leq & 0 \\ x & + & 2y & \leq & 6 \\ -x & - & y & \leq & -2 \\ x & - & y & \leq & 3 \\ & - & y & \leq & 0 \end{cases}$$

Please give the vertices of P.

b) Given a 2-dimensional rectangle R which is defined by the convex hull of the points $(0,0), (0,1), (1,1), (1,0)$. Please give the vertices of the convex hull of R and P (given in the previous exercise), and the linear inequalities which define it.

Solution:

a) The given inequalities form the following polytope:

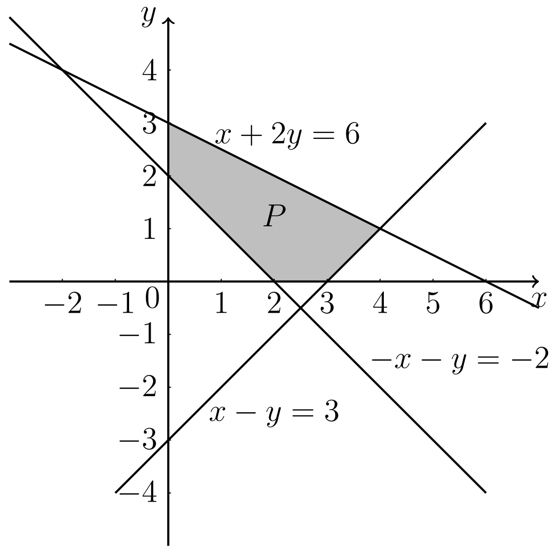


Figure 1: The polyhedron P

b) We use the vertices from the previous exercise and the given vertices to compute the convex hull: The vertices of the convex hull are $(0,0),(0,3),(4,1),(3,0)$, and

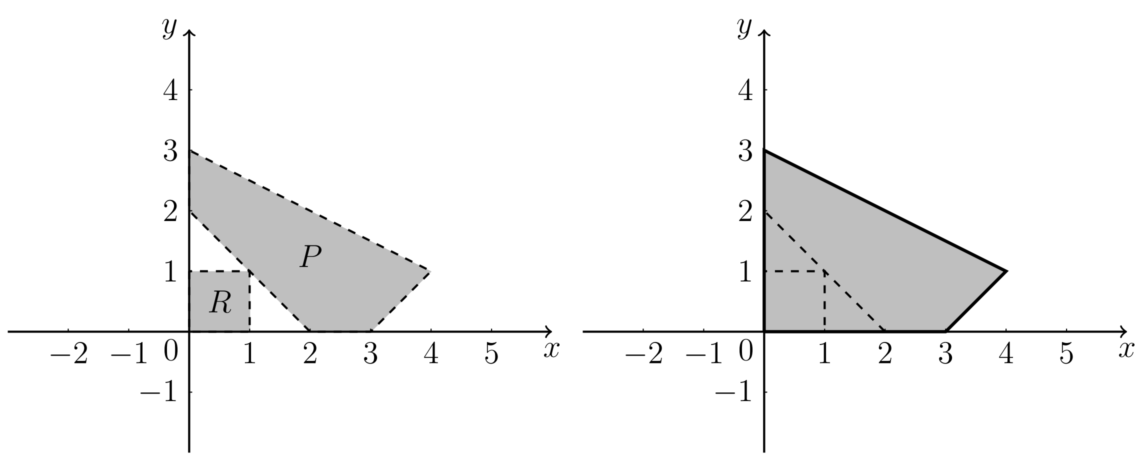


Figure 2: The convex hull of P and R

$conv(P, R)$ can be defined by the inequalities

$$-x \leq 0, -y \leq 0, x + 2y \leq 6, x - y \leq 3$$

Exercise 5

Why is the choice of the state set representation crucial?

Solution:
The choice of the used state set representation has a strong impact on the computational time and space required for the reachability analysis. On the other hand it also has an effect on the overappoximation error we introduce by the abstraction. In general this is a trade of between complexity and precision.