1) Discrete $\xrightarrow{MC}$ Answer

2) Timed automata — Answer
↓ abstraction ↑
Discrete $\xrightarrow{MC}$ Answer

3) Rectangular automata, initialized — Answer
↓ Trasform ↑
⋮
Timed automaton $\xrightarrow{MC}$ Answer

4) Linear hybrid automata I } Exact for bounded reachability

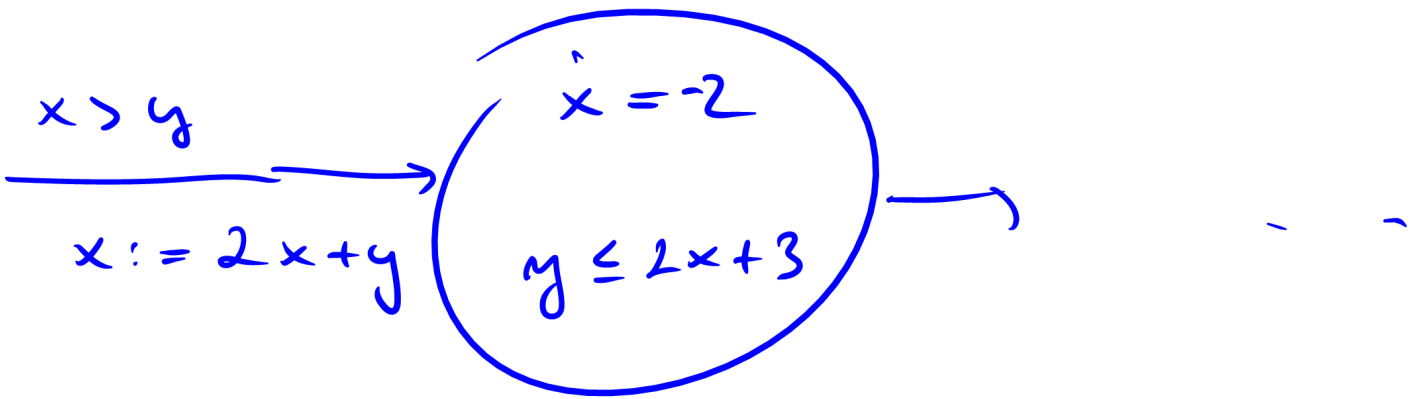
$\dot{x} = c$

I+C) Linear II } over-approximative

Exact

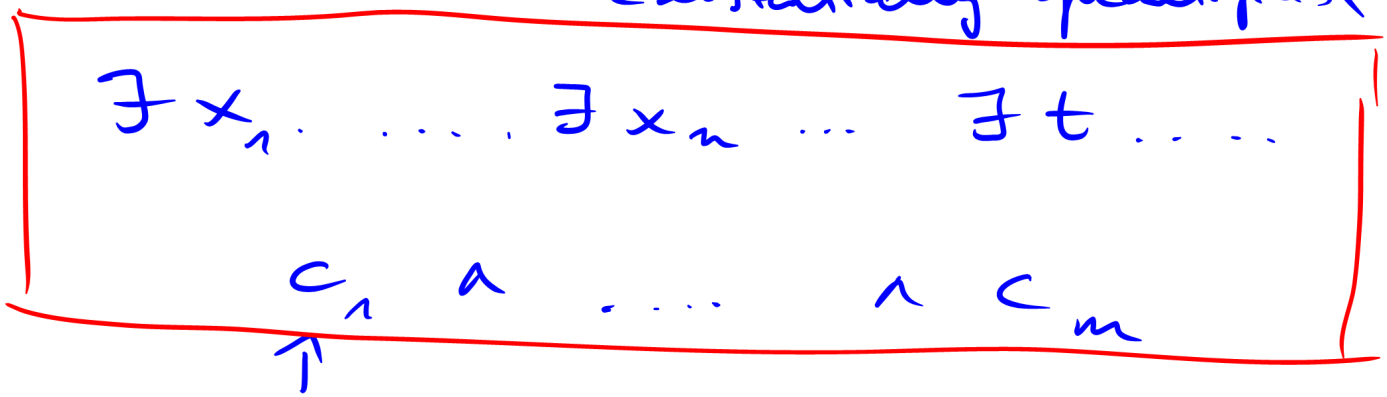
LHA I

Example :



Bounded Reachability for LHAI can be encoded by formulas of linear real arithmetic.

existentially quantified (LRA)



linear real arith. constraint

Solvable in polynomial time

$$\frac{\exists e = (\ell, a, g, r, \ell') \in Edge \quad \nu \models g \quad \nu' \models \exists w(\ell') \,.\, \nu \oplus \nu'' \models r}{(\ell, \nu) \xrightarrow{a} (\ell', \nu')}$$

Rule discr

g : formula over Var

r : -ll- over Var ∪ Var'

I : LRA formula over Var

$e = (\ell, a, g, r, \ell') \in Edge$

$\{x_1, \dots, x_n\} = Var$

$D_e^+(I)$: LRA formula over Var

$$D_e^+(I) = \exists x_1^{pre} \dots \exists x_n^{pre} \; I\left[\frac{x_1^{pre}, \dots, x_n^{pre}}{x_1, \dots, x_n}\right] \wedge g\left[\frac{x_1^{pre}, \dots, x_n^{pre}}{x_1, \dots, x_n}\right] \wedge \exists w(\ell) \wedge r\left[\frac{x_1^{pre}, \dots, x_n^{pre}, x_1, \dots, x_n}{x_1, \dots, x_n, x_1', \dots, x_n'}\right]$$

$$g\left[\frac{\nu(Var)}{Var}\right] \equiv true$$

$\Updownarrow$ I

$\nu \models g$

$$x_1^{pre}, \dots, x_n^{pre} \xrightarrow{a} x_1, \dots, x_n$$

$\Updownarrow$

$\exists$ step + time

+ Quantifier elimination for conjunctive formulas

I : $\exists x \ldots x \ldots = \ldots$ LRA eq.

$\downarrow$

$x = T$

$\downarrow$

Replace x by T

II Fourier-Motzkin

$\exists x \ldots \quad x \ldots \leq \ldots \quad \wedge$

$\ldots x \ldots \leq \quad \wedge$

$\ldots \quad \leq x \ldots$

$\downarrow$

$\exists x \quad \forall i,j \quad l_i \leq u_j$

$l_i \leq x \leq u_j \iff l_i \leq u_j$

conjunction !

$\exists x \ldots \rightarrow x = y \wedge x = y+1$

$\exists x \ldots \not\rightarrow x = y \vee x = y+1$

Modeling and Analysis of Hybrid Systems

Linear hybrid automata I

Prof. Dr. Erika Ábrahám

Informatik 2 - LuFG Theory of Hybrid Systems
RWTH Aachen University

Szeged, Hungary, 27 September - 06 October 2017

Literature

Alur et al.: The algorithmic analysis of hybrid systems

Theoretical Computer Science, 138(1):3–34, 1995

Linear terms, constraints and formulas

- A linear term e over a set Var of variables is of the form

$$e ::= c \mid c \cdot x \mid e + e$$

where $x \in Var$ is a variable and c stays for an integer (rational) constant.

Example: $x_1 + 2x_2 + 5x_3$ is a linear term over $Var = \{x_1, x_2, x_3\}$.

- A linear constraint t over Var is an (in)equality

$$t ::= e \sim 0$$

with $\sim \in \{>, \geq, =, \leq, <\}$ and e a linear term over Var.

Example: $x_1 + 2x_2 + (-2) \geq 0$ is a linear constraint over $Var = \{x_1, x_2\}$. We sometimes deviate from this normal form and write, e.g., $x_1 + 2x_2 \geq 2$.

Linear terms, constraints and formulas

- A conjunctive linear formula φ over Var is defined by the following grammar:

$$\varphi ::= t \mid \varphi \wedge \varphi \mid \exists x.\ \varphi$$

with t a linear constraint over Var and $x \in Var$ a variable. Let Φ_X be the set of all conjunctive linear formulas with free (non-quantified) variables from $X \subseteq Var$.

Example: $\exists t.\ \exists x_1.\ x_1 \geq 0 \wedge x_1' = x_1 + 2t \wedge t \geq 0$ is a conjunctive linear formula over $\{x_1, x_1', t\}$, with free (non-quantified) variables $\{x_1'\}$.

Regions

- A region over Var is a pair $R = (l, \varphi) \in Loc \times \Phi_{Var}$ of a location and a conjunctive linear formula with free variables from Var.

  Example: $(l, \exists t.\ \exists x_1.\ x_1 \geq 0 \wedge x_1' = x_1 + 2t \wedge t \geq 0)$ is a region over $\{x_1'\}$.
- The intersection of two regions $R^1 = (l_1, \varphi_1)$ and $R^2 = (l_2, \varphi_2)$ from $Loc \times \Phi_{Var}$ is defined as $R^1 \hat{\cap} R^2 = \emptyset$ if $l_1 \neq l_2$ and

$$R^1 \hat{\cap} R^2 = (l_1, \varphi_1 \wedge \varphi_2)$$

  otherwise.
- The intersection of two sets of regions $P^1, P^2 \subseteq Loc \times \Phi_{Var}$ is defined as

$$P^1 \hat{\cap} P^2 = \{(l, \varphi_1 \wedge \varphi_2) \mid (l, \varphi_1) \in P^1, (l, \varphi_2) \in P^2\}\ .$$

- We define other operations on regions like union etc. similarly, and extend them to sets of regions the natural way.

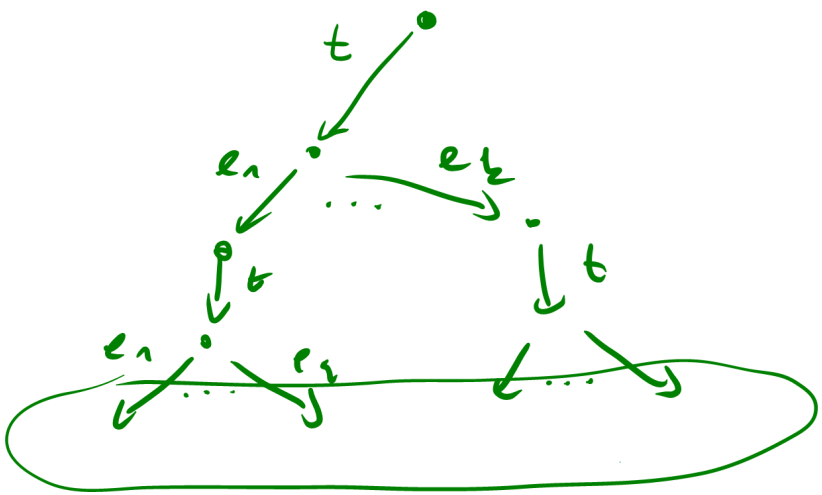
LLA formula $\longleftrightarrow$ valuation set

$x \geq 0 \quad \longleftrightarrow \quad \{ v \in V \mid v(x) \geq 0 \}$

$\wedge \quad \vee \quad \neg \quad \longleftrightarrow \quad \cap \quad \cup \quad \overline{\cdot}$

$\rightarrow \quad \leftrightarrow \quad \subseteq \quad =$

substitution $\quad \in$



Substitution

- Assume a set Var of variables, a linear formula $\varphi \in \Phi_{Var}$ over Var, a variable $x \in Var$, and a linear term e over Var. The substitution $\varphi[e/x]$ replaces each free occurrence of x in φ by e.

  Example: $(x + 2y \leq 0)[5/y] = x + 2 \cdot 5 \leq 0$

  Example: $(\exists y.\ x + 2y \leq 0)[5/y] = x + 2y \leq 0$
- We write $\varphi[e_1, \ldots, e_n/x_1, \ldots, x_n]$ for the simultaneous substitution of e_i for x_i, $i = 1, \ldots, n$.
- For $Var = \{x_1, \ldots, x_n\}$ we will also use a primed variable set $Var' = \{x'_1, \ldots, x'_n\}$ and write short $\varphi[Var'/Var]$ for $\varphi[x'_1, \ldots, x'_n/x_1, \ldots, x_n]$.

Linear terms, constraints and formulas

The semantics of linear terms, constraints and formulas over $Var = \{x_1, \ldots, x_n\}$ in the context of a valuation $\nu \in V_{Var}$ (i.e., $\nu : Var \to \mathbb{R}$) is as usual (we use the same notation for the syntax and the semantics of constants and operators):

$$
\begin{array}{lcl}
\nu(c) & \equiv & c \\
\nu(c \cdot x) & \equiv & c \cdot \nu(x) \\
\nu(e_1 + e_2) & \equiv & \nu(e_1) + \nu(e_2) \\
\nu(e \sim 0) & \equiv & \nu(e) \sim 0 \\
\nu(\varphi_1 \wedge \varphi_2) & \equiv & \nu(\varphi_1) \textit{ and } \nu(\varphi_2) \\
\nu(\exists x.\ \varphi) & \equiv & \textit{exists } v \in \mathbb{R} \textit{ such that } \nu(\varphi[v/x]) \textit{ holds}
\end{array}
$$

Linear terms, constraints and formulas

The solution set $Sat(\varphi)$ of a linear formula φ over Var is the set of all valuations $\nu \in V_{Var}$ that make φ true:

$$Sat(\varphi) = \{\nu \in V_{Var} \mid \nu \models \varphi\}$$

The solution set $Sat(R)$ of a region $R = (l, \varphi) \in Loc \times \Phi_{Var}$ over Loc and Var is

$$Sat((l, \varphi)) = \{(l, \nu) \in Loc \times V_{Var} \mid \nu \in Sat(\varphi)\}$$

The solution set $Sat(P)$ of a set of regions P over Loc and Var is

$$Sat(P) = \cup_{R \in P} Sat(R) \ .$$

Two region sets $P_1, P_2 \subseteq Loc \times \Phi_{Var}$ are equivalent, written $P_1 \hat{=} P_2$, iff

$$Sat(P_1) = Sat(P_2) \ .$$

We define similarly the inclusion $P_1 \hat{\subseteq} P_2$ iff

$$Sat(P_1) \subseteq Sat(P_2) \ .$$

Linear hybrid automata I

A linear hybrid automaton is a hybrid automaton $\mathcal{H} = (Loc, Var, Lab, Edge, Act, Inv, Init)$ which can be represented by a tuple $\mathtt{H} = (Loc, Var, Lab, \mathtt{Edge}, \mathtt{Act}, \mathtt{Inv}, \mathtt{Init})$ satisfying the following:

- The finite set $\mathtt{Edge} \subseteq Loc \times \Phi_{Var} \times \Phi_{Var \cup Var'} \times Loc$ defines the set of edges $Edge = \{(l, a, \mu_e, l') \mid e = (l, a, \mathtt{Guard}_e, \mathtt{Reset}_e, l') \in \mathtt{Edge}\}$, where the transition relation μ_e is given as

  $$\mu_e = \{(\nu, \nu') \in V_{Var} \times V_{Var} \mid \nu \in Sat(\mathtt{Guard}_e) \text{ and } (\nu \oplus \nu') \in Sat(\mathtt{Reset}_e)\}$$

  with $\nu \oplus \nu' \in V_{Var \cup Var'}$ such that $(\nu \oplus \nu')(x) = \nu(x)$ for $x \in Var$ and $(\nu \oplus \nu')(x') = \nu'(x)$ for $x' \in Var'$.

- The set $\mathtt{Act} \subseteq Loc \times \Phi_{Var}$ contains for each location $l \in Loc$ exactly one region $(l, \mathtt{Act}_l) \in \mathtt{Act}$ whose second component is of the form $\mathtt{Act}_l = \bigwedge_{i=1}^{n} x_i + k_{l,i}^{lower} t \leq x_i' \wedge x_i' \leq x_i + k_{l,i}^{upper} t$, defining the activities

  $$Act(l) = \{f : \mathbb{R}_{\geq 0} \to V_{Var} \mid \dot{f}_{x_i} \in [k_{l,i}^{lower}, k_{l,i}^{upper}] \text{ for all } i \in \{1, \ldots, n\}\},$$

  where $f_{x_i} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $f_{x_i}(t) = f(t)(x_i)$ for all $t \in \mathbb{R}_{\geq 0}$.

Linear hybrid automata I (cont.)

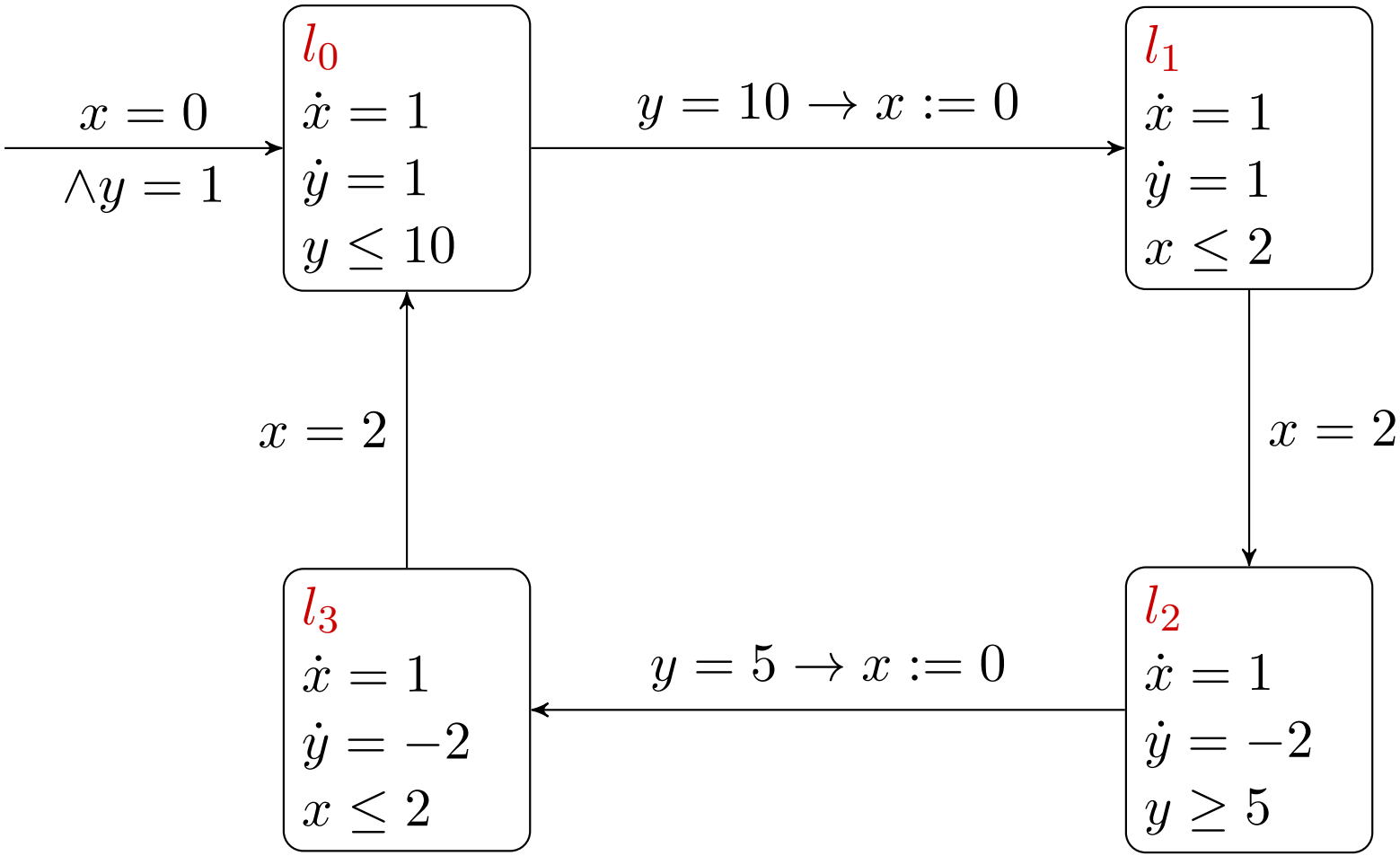
- The set $\texttt{Inv} \subseteq Loc \times \Phi_{Var}$ contains for each location $l \in Loc$ exactly one region $(l, \texttt{Inv}_l) \in \texttt{Inv}$ such that

$$Inv(l) = Sat(\texttt{Inv}_l) \ .$$

- $\texttt{Init} \subseteq Loc \times \Phi_{Var}$ is a finite set of regions specifying the initial states by

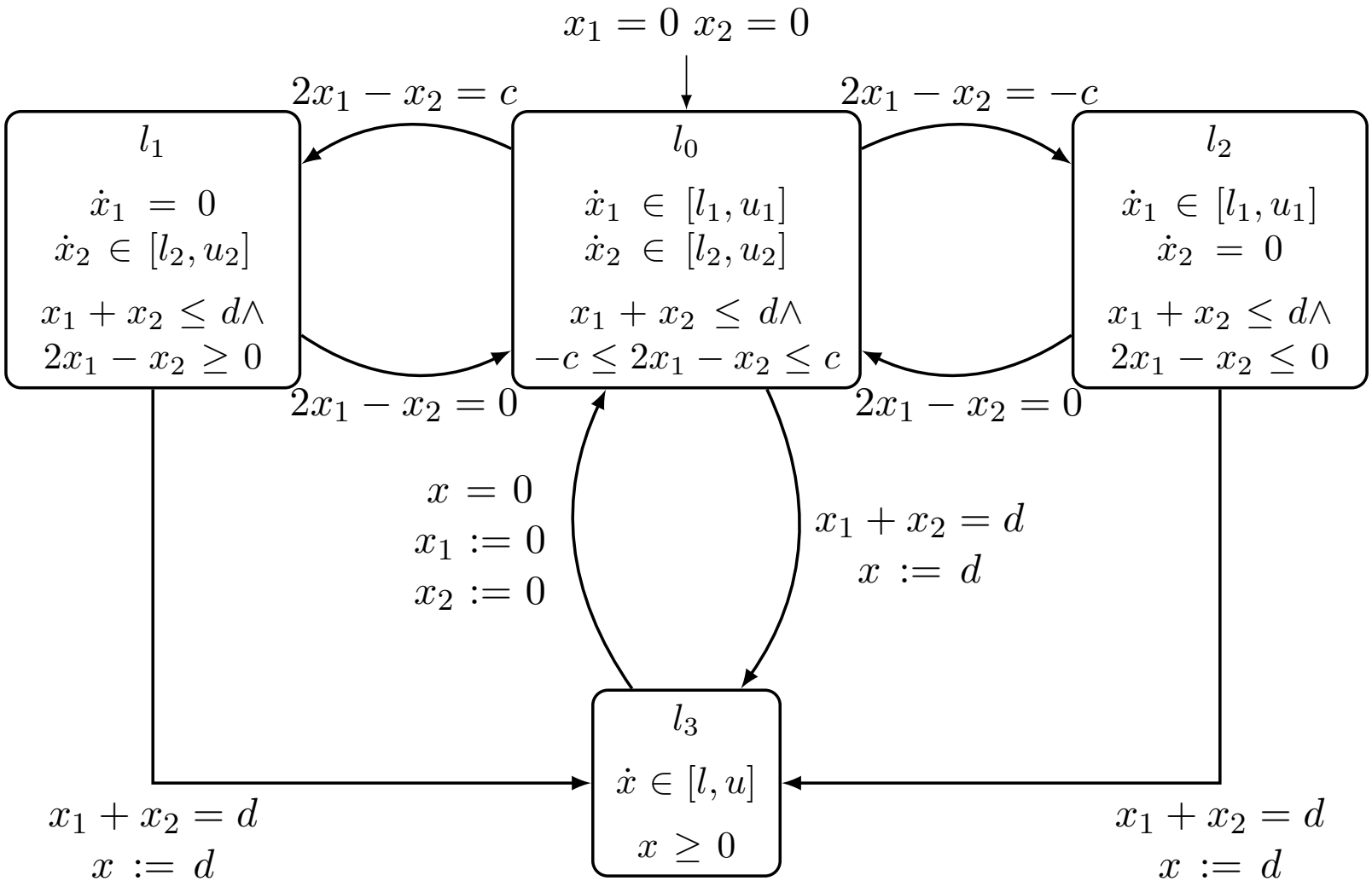
$$Init = \bigcup_{(l,\varphi) \in \texttt{Init}} \{(l, \nu) \mid \nu \in Sat(\varphi)\} \ .$$

Water-level monitor



Mixer of fluids

$$0 < l_1 < u_1,\ 0 < l_2 < u_2,\ l \leq u < 0,\ d > 0,\ c > 0$$



Reminder: Semantics of hybrid automata

$$\frac{(l, a, \mu, l') \in Edge \quad (\nu, \nu') \in \mu \quad \nu' \in Inv(l')}{(l, \nu) \xrightarrow{a} (l', \nu')} \text{ Rule } \texttt{Discrete}$$

$$\frac{\begin{array}{l} f \in Act(l) \quad f(0) = \nu \quad f(t) = \nu' \\ t \geq 0 \quad \forall 0 \leq t' \leq t . f(t') \in Inv(l) \end{array}}{(l, \nu) \xrightarrow{t} (l, \nu')} \text{ Rule } \texttt{Time}$$

Forward analysis

- Given a set of initial states $Init \subseteq \Sigma$, we want to compute the set of all states which are reachable from $Init$:

$$Reach^{+}(Init) = \{\sigma' \in \Sigma \mid \exists \sigma \in Init.\ \sigma \rightarrow^{*} \sigma'\} \ .$$

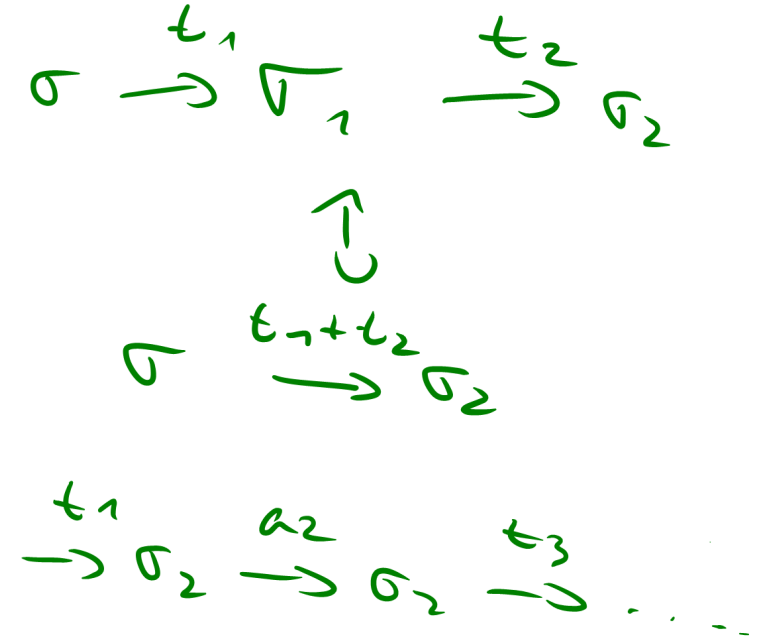
- Given a set of initial states $Init \subseteq \Sigma$, we want to compute the set of all states which are reachable from $Init$:

$$Reach^{+}(Init) = \{\sigma' \in \Sigma \mid \exists \sigma \in Init.\ \sigma \rightarrow^{*} \sigma'\} \ .$$

- More specifically, we want to check whether the reachable region intersects with a set of bad (unsafe) states.

Forward reachability analysis

```
method forward_reach(
    hybrid automaton representation H = (Loc, Var, Lab, Edge, Act, Inv, Init),
    region set P^bad) {
  //start from the time successors of initial regions
  P^0 := T+(Init ∩̂ Inv);
  if (Sat(P^0 ∩̂ P^bad) ≠ ∅) return unsafe;
  i := 0;
  while P^i ≠ ∅ {
    //compute P^{i+1} := T+(D+(P^i))
    P^{i+1} := ∅;
    for each (R = (l, φ) ∈ P^i) {
      for each e = (l, ..., l') ∈ Edge {
        R' := T+_{l'}(D+_e(R));
        if (Sat({R'} ∩̂ P^bad) ≠ ∅) return unsafe;
        else if (not {R'} ⊆̂ ⋃̂_{j=0}^{i} P^j)
          P^{i+1} := P^{i+1} ∪̂ {R';
      }
    }
    i := i+1;
  }
  return safe;
}
```



① $\{(\ell_1, \varphi_1), \ldots, (\ell_k, \varphi_k)\} = \mathrm{Init}$

② $\mathrm{Init} \hat{\cap} \mathrm{Inv}$

$\{(\ell_1, \varphi_1), \ldots, (\ell_\ell, \varphi_\ell)\} \hat{\cap} \{(\ell_1, \varphi_1'), \ldots, (\ell_k, \varphi_\ell')\}$

$= \{(\ell_1, \varphi_1 \wedge \varphi_1'), \ldots, (\ell_\ell, \varphi_\ell \wedge \varphi_\ell')\}$

③ $T^+(\mathrm{Init} \hat{\cap} \mathrm{Inv}) = \{(\ell_1, \varphi_1''), \ldots, (\ell_\ell, \varphi_\ell'')\} =: P_0$

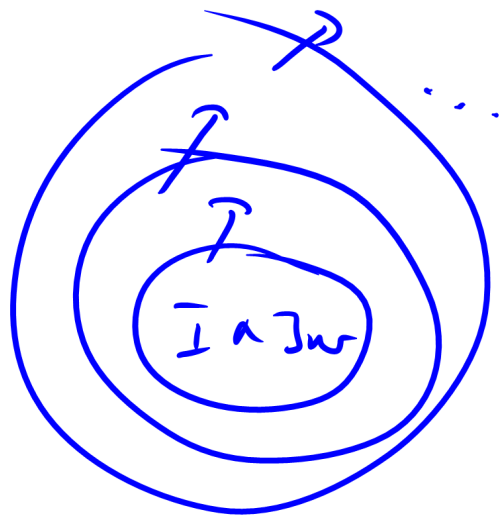
④ $D^+(P_0)$

⑤ $T^+(D^+(P_0)) =: P_1$

⑥ $D^+(P_1)$

⑦ $T^+(D^+(P_1)) =: P_2$

$P_1 \hat{\vee} \ldots \hat{\vee} P_i \supseteq P_{i+1}$



One-step reachability under time steps

- We define the forward time closure $\mathcal{T}_l^+(\varphi)$ of a formula $\varphi \in \Phi_{Var}$ at $l \in Loc$ as

$$\mathcal{T}_l^+(\varphi) = \exists x_{pre}.\ \exists t.\ t \geq 0 \wedge \varphi[x_{pre}/x] \wedge \mathtt{Act}_l[x_{pre}, x/x, x'] \wedge \mathtt{Inv}_l$$

- Region $R = (l, \varphi) \in Loc \times \Phi_{Var}$:

$$\mathcal{T}_l^+(R) = (l, \mathcal{T}_l^+(\varphi))$$

- Set of regions $P \subseteq Loc \times \Phi_{Var}$:

$$\mathcal{T}^+(P) = \{\mathcal{T}_l^+(R) \mid R = (l, \varphi) \in P\}$$

One-step reachability under discrete steps

- We define the postcondition $\mathcal{D}_e^+(\varphi)$ of a formula $\varphi \in \Phi_{Var}$ with respect to an edge $e = (l, \texttt{Guard}_e, \texttt{Reset}_e, l')$ as

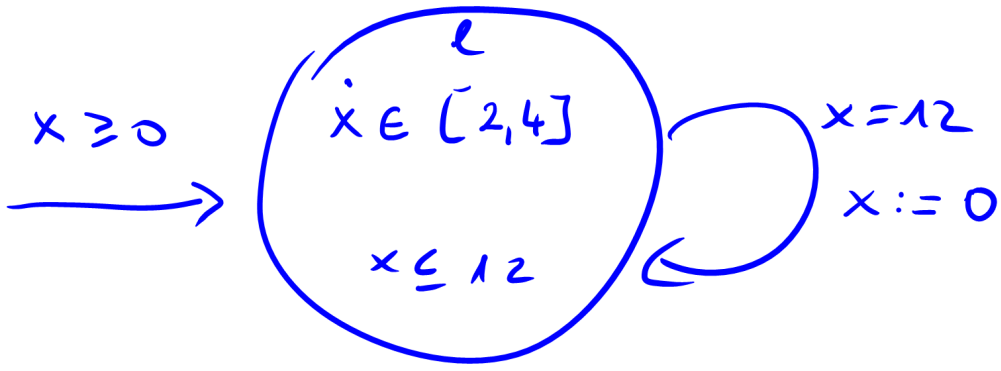
$$\mathcal{D}_e^+(\varphi) = \exists x_{pre}.\ \varphi[x_{pre}/x] \wedge \texttt{Guard}_e[x_{pre}/x] \wedge \texttt{Reset}_e[x_{pre}, x/x, x'] \wedge \texttt{Inv}_{l'}.$$

- Region $R = (l, \varphi) \in Loc \times \Phi_{Var}$:

$$\mathcal{D}_e^+(R) = (l', \mathcal{D}_e^+(\varphi))$$

- Set of regions $P \subseteq Loc \times \Phi_{Var}$:

$$\mathcal{D}^+(P) = \{\mathcal{D}_e^+(R) \mid R = (l, \varphi) \in P,\ e = (l, \texttt{Guard}_e, \texttt{Reset}_e, l') \in \texttt{Edge}\}$$



$I = \{(\ell, x \geq 0)\}$ $\quad Inv = \{(\ell, x \leq 12)\}$

$I \hat{\cap} Inv = \{(\ell, 0 \leq x \leq 12)\}$

$T^+(I \hat{\cap} Inv) = \{(\ell, \varphi_0)\}$

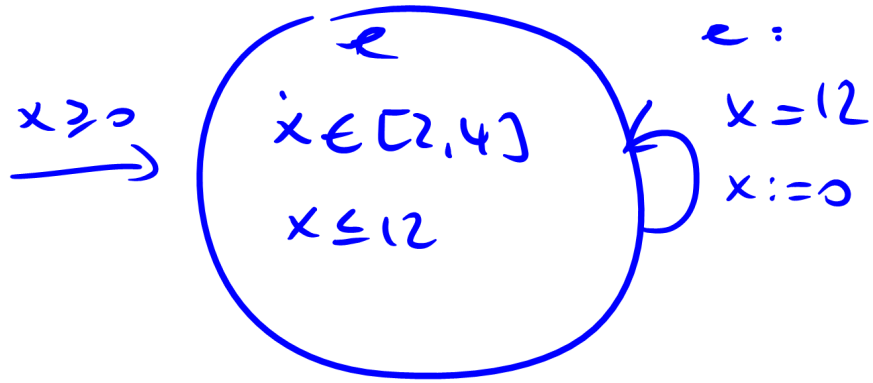
$\varphi_0 : \exists x^{pre} \exists t.\ 0 \leq x^{pre} \leq 12 \wedge t \geq 0 \wedge x^{pre} + 2t \leq x \leq x^{pre} + 4t \wedge x \leq 12$

$x \leq 12 \equiv$ | $0 \leq x^{pre} \leq 12 \wedge t \geq 0 \wedge x - 4t \leq x^{pre} \leq x - 2t \wedge x \leq 12$ |

$\exists t.$

$\equiv \cancel{0 \leq 12} \wedge 0 \leq x - 2t \wedge x - 4t \leq 12 \wedge \cancel{x - 4t \leq x - 2t} \wedge t \geq 0 \wedge x \leq 12 \equiv t \leq \frac{1}{2}x \wedge \frac{1}{4}x - 3 \leq t \wedge 0 \leq t \wedge x \leq 12$

$\equiv \frac{1}{4}x - 3 \leq \frac{1}{2}x \wedge 0 \leq \frac{1}{2}x \wedge x \leq 12 \equiv 0 \leq x \leq 12$



$\varphi_0 = \{(\ell_1, 0 \leq x \leq 12)\}$

$D^+(\varphi_0) = \{(\ell_1, \varphi_1)\}$ $\quad \varphi_0' = D_e^+(0 \leq x \leq 12) \equiv$

$\exists\, x^{pre}.\ 0 \leq x^{pre} \leq 12 \wedge x^{pre} = 12 \wedge x = 0 \wedge x \leq 12 \equiv$

$0 \leq 12 \leq 12 \wedge x = 0 \wedge x \leq 12 \equiv \boxed{x = 0}$

$T^+(D^+(\varphi_0)) = \{(\ell_1, T^+(\varphi_0'))\} = \{T^+(\ell_1, x = 0)\} \equiv$

$\exists\, x^{pre}\ \exists t.\ t \geq 0 \wedge x^{pre} = 0 \wedge x^{pre} + 2t \leq x \leq x^{pre} + 4t \wedge x \leq 12$

$\equiv \exists t.\ t \geq 0 \wedge 2t \leq x \leq 4t \wedge x \leq 12$

$\equiv \exists t.\ t \geq 0 \wedge t \leq \frac{x}{2} \wedge \frac{1}{4}x \leq t \wedge x \leq 12$

$\equiv 0 \leq \frac{1}{2}x \wedge \frac{1}{4}x \leq \frac{1}{2}x \wedge x \leq 12 \equiv 0 \leq x \leq 12$

$T^+(D^+(\varphi_0)) \equiv \varphi_1$

Backward analysis

- Given a set of target states $B \subseteq \Sigma$, we want to compute the set of all states from which a state in B is reachable:

$$Reach^{-}(B) = \{\sigma \in \Sigma \mid \exists \sigma' \in B.\ \sigma \rightarrow^{*} \sigma'\}\ .$$

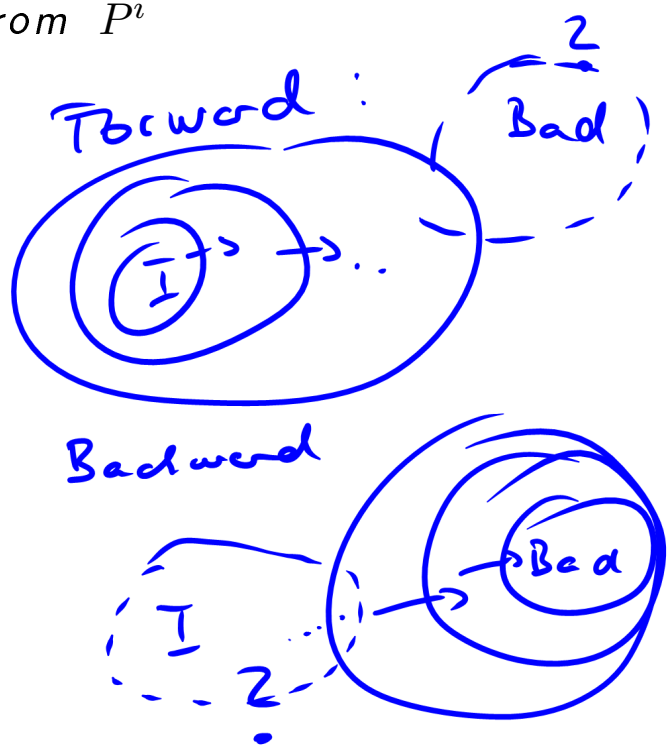
- Given a set of target states $B \subseteq \Sigma$, we want to compute the set of all states from which a state in B is reachable:

$$Reach^{-}(B) = \{\sigma \in \Sigma \mid \exists \sigma' \in B.\ \sigma \rightarrow^{*} \sigma'\}\ .$$

- More specifically, we want to check whether the set of backward reachable states intersects with a set of initial states.

Backward reachability analysis

```
method backward_reach() {
  i := 0;
  P^0 := {T_l^-(R) | R = (l, φ) ∈ P^bad ∩̂ Inv}; //time predec. of bad regions
  if (Sat(P^0 ∩̂ Init) ≠ ∅) return unsafe;
  while P^i ≠ ∅ {
    //compute time predecessors of
    //discrete predecessors of all regions from P^i
    for each (R = (l, φ) ∈ P^i) {
      for each e = (l', ..., l) ∈ Edge {
        R' := T_l'^-(D_e^-(R));
        if (Sat({R'} ∩̂ Init) ≠ ∅) return unsafe;
        else if (not {R'} ⊆̂ ⋃_{j=0}^{i} P^j)
          P^{i+1} := P^{i+1} ∪ {R'};
      }
    }
    i := i+1;
  }
  return safe;
}
```



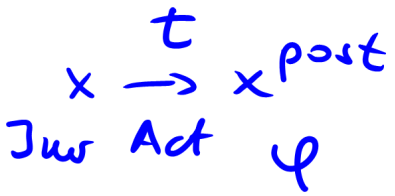
One-step reachability under time steps

- We define the backward time closure $\mathcal{T}_l^-(\varphi)$ of a formula $\varphi \in \Phi_{Var}$ at $l \in Loc$ as

$$\mathcal{T}_l^-(\varphi) = \exists x_{post}.\ \exists t.\ t \geq 0 \wedge \varphi[x_{post}/x] \wedge \mathtt{Act}_l[x, x_{post}/x, x'] \wedge \mathtt{Inv}_l\ .$$

- Region $R = (l, \varphi) \in Loc \times \Phi_{Var}$:

$$\mathcal{T}_l^-(R) = (l, \mathcal{T}_l^-(\varphi))$$



- Set of regions $P \subseteq Loc \times \Phi_{Var}$:

$$\mathcal{T}^-(P) = \{\mathcal{T}_l^-(R) \mid R = (l, \varphi) \in P\}$$

One-step reachability under discrete steps

- We define the precondition $\mathcal{D}_e^-(\varphi)$ of a formula $\varphi \in \Phi_{Var}$ with respect to an edge $e = (l, \texttt{Guard}_e, \texttt{Reset}_e, l')$ as

$$\mathcal{D}_e^-(\varphi) = \exists x_{post}.\ \varphi[x_{post}/x] \wedge \texttt{Guard}_e \wedge \texttt{Reset}_e[x, x_{post}/x, x'] \wedge \texttt{Inv}_l.$$

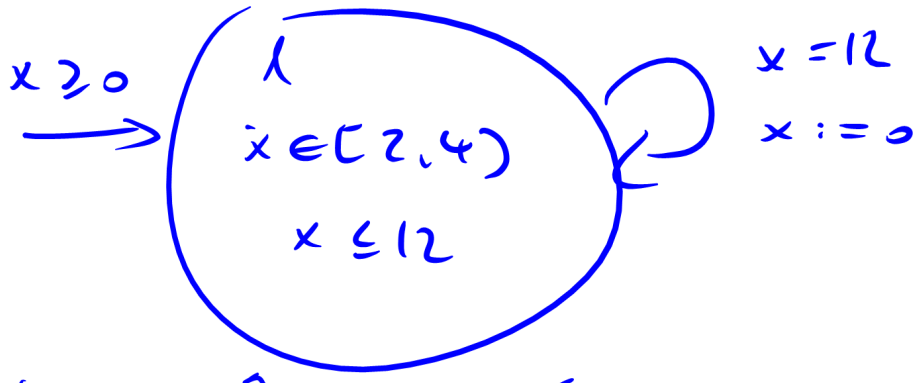
- Region $R = (l', \varphi) \in Loc \times \Phi_{Var}$:

$$\mathcal{D}_e^-(R) = (l, \mathcal{D}_e^-(\varphi))$$

x -> x post
Guard Reset
Inv

- Set of regions $P \subseteq Loc \times \Phi_{Var}$:

$$\mathcal{D}^-(P) = \{\mathcal{D}_e^-(R) \mid R = (l', \varphi) \in P,\ e = (l, \texttt{Guard}_e, \texttt{Reset}_e, l') \in \texttt{Edge}\}$$



$\varphi_{bad} = \{ (l, x < 0) \}$

$x \xrightarrow{t} x^{post}$

$\varphi_{bad} \hat{\cap} Inv = \{ (l, x < 0) \}$

$T^{-}(\varphi_{bad} \hat{\cap} Inv) =$

$\exists x^{post}\ \exists t.\ t \geq 0 \wedge x^{post} < 0 \wedge x + 2t \leq x^{post} \leq x + 4t \wedge x \leq 12$

$\exists t.$

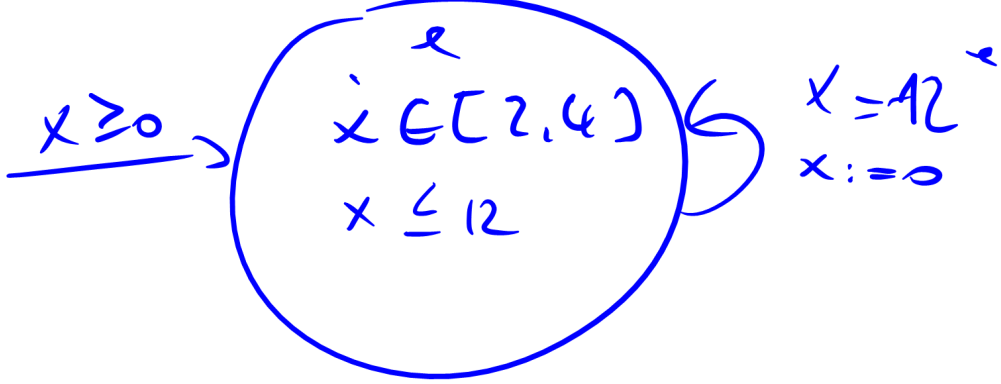
$\equiv x + 2t < 0 \wedge \underbrace{x + 2t \leq x + 4t}_{0 \leq t} \wedge t \geq 0 \wedge x \leq 12$

$\exists t.$

$\equiv x + 2t < 0 \wedge 0 \leq t \wedge x \leq 12 \equiv \exists t.\ t \in -\frac{1}{2}x \wedge 0 \leq t \wedge x \leq 12$

$\equiv 0 < -\frac{1}{2}x \wedge x \leq 12 \equiv x < 0$

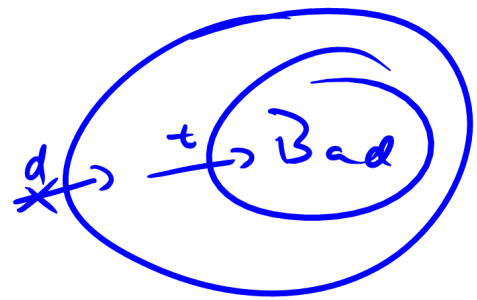
$\varphi_0 = \{ (l, x < 0) \}$



$\exists x^{post}. x \rightarrow x^{post}$

$\varphi_0 = \{(e_1, x < 0)\}$

$D^-(\varphi_0) = D_e^-(\varphi_0) \equiv \exists x^{post}. x^{post} < 0 \wedge x^{post} = 0 \wedge \ldots$



$\sigma_0 \xrightarrow{d} \sigma_1 \xrightarrow[\checkmark]{t} \sigma_2$

Quantifier elimination

- Problem 1: Formula size increases steeply
- Problem 2: In the presence of quantifiers, computing operations (inclusion, intersection etc.) on regions is non-trivial
- Solution: Therefore, we eliminate quantifiers: given a conjunctive linear formula $\exists x.\ \varphi \in \Phi_X$, we compute another conjunctive linear formula $\varphi' \in \Phi_{X \setminus \{x\}}$ that does not contain x such that

$$\mathit{Sat}(\exists x.\ \varphi) = \mathit{Sat}(\varphi') \ .$$

- Technique: Gauß and Fourier-Motzkin variable elimination

Linear real arithmetic: Gauß method for equations

- Assume that φ is of the form $\exists x_n.\ \varphi'' \wedge \sum_{k=1}^{n} a_k \cdot x_k = b$ with $a_n \neq 0$.

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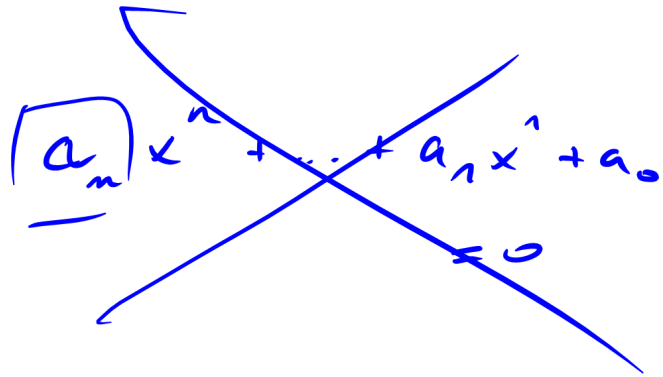

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- Replace x_n by β in φ''.
- This substitutiton leads to an equisatisfiable problem in $n-1$ variables:

$$\mathit{Sat}(\exists x_n.\ \varphi'' \wedge \sum_{k=1}^{n} a_k \cdot x_k = b) \quad = \quad \mathit{Sat}(\varphi''[\beta/x_n]) \quad (\text{for } a_n \neq 0).$$

Linear real arithmetic: Fourier-Motzkin for inequalities

- Discovered in 1826 by Fourier, re-discovered by Motzkin in 1936
- Given:

$$\exists x_n. \bigwedge_{1 \leq i \leq m} \sum_{1 \leq j \leq n} a_{ij} x_j \leq b_i$$

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$\ell \le x \le u$

$\downarrow$

$\ell \le u$

$$(a) \quad \overset{a_{in} > 0}{\Rightarrow} \quad x_n \le \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \quad =: \beta_l \qquad \text{upper bound}$$

$$(b) \quad \overset{a_{in} < 0}{\Rightarrow} \quad x_n \ge \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \quad =: \beta_u \qquad \text{lower bound}$$

Example for upper and lower bounds

Category for x_1?

$$
\begin{aligned}
&(1) \quad x_1 - x_2 \leq 0 \\
&(2) \quad x_1 - x_3 \leq 0 \\
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Eliminating unbounded variables

- Iteratively remove variables that are not bounded in both ways (and all the constraints that use them).
- The new problem has a solution iff the old problem has one!

$$\begin{aligned} 8x &\geq 7y \\ x &\geq 3 \\ y &\geq z \\ z &\geq 10 \\ 20 &\geq z \end{aligned}$$

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Fourier-Motzkin variable elimination

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- For each such pair, add the constraint

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Fourier-Motzkin: Example

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we eliminate x_3

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---

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---

we eliminate x_3

(7) $1 \leq 0$ (from 4,5)

$\rightarrow$ Contradiction (the system is UNSAT)

Complexity

- Worst-case complexity:

$$m \rightarrow m^2$$

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- Heavy!

$$\dot{x} = Ax + Bu$$