



Modeling and Analysis of Hybrid Systems

Series 5

Exercise 1

A gas burner is a device to generate a flame to heat up products using a gaseous fuel. We assume there is a gas burner, such that

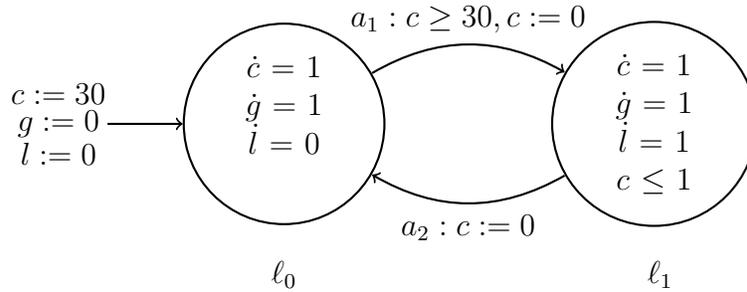
- (a) any leakage of it can be detected and stopped within **1** second,
- (b) it does not leak initially, and
- (c) it never leaks for at least **30** seconds after a leakage has been stopped.

The gas burner records the cumulative *leakage time* l and the *total elapsed time* g .

- (1) Please model the gas burner by a hybrid automaton and try to keep it as simple as possible.
- (2) Is it possible to have a (cumulative) 2-second leakage time in 70 seconds? If so, please give a sample execution.

Solution:

1. The gas burner can be modeled by the following hybrid automaton. It contains



3 clocks, the local clock c , the global time g and the cumulative leak time l .

2. Yes, it is possible. We give one example execution as follows.

$$\begin{aligned}
 (\ell_0, (30, 0, 0)) &\xrightarrow{0} (\ell_0, (30, 0, 0)) \xrightarrow{a_1} (\ell_1, (0, 0, 0)) \xrightarrow{1} (\ell_1, (1, 1, 1)) \xrightarrow{a_2} (\ell_0, (0, 1, 1)) \\
 &\xrightarrow{32} (\ell_0, (32, 33, 1)) \xrightarrow{a_1} (\ell_1, (0, 33, 1)) \xrightarrow{1} (\ell_1, (1, 34, 2)) \xrightarrow{a_2} (\ell_0, (0, 34, 2)) \xrightarrow{36} (\ell_0, (36, 70, 2))
 \end{aligned}$$

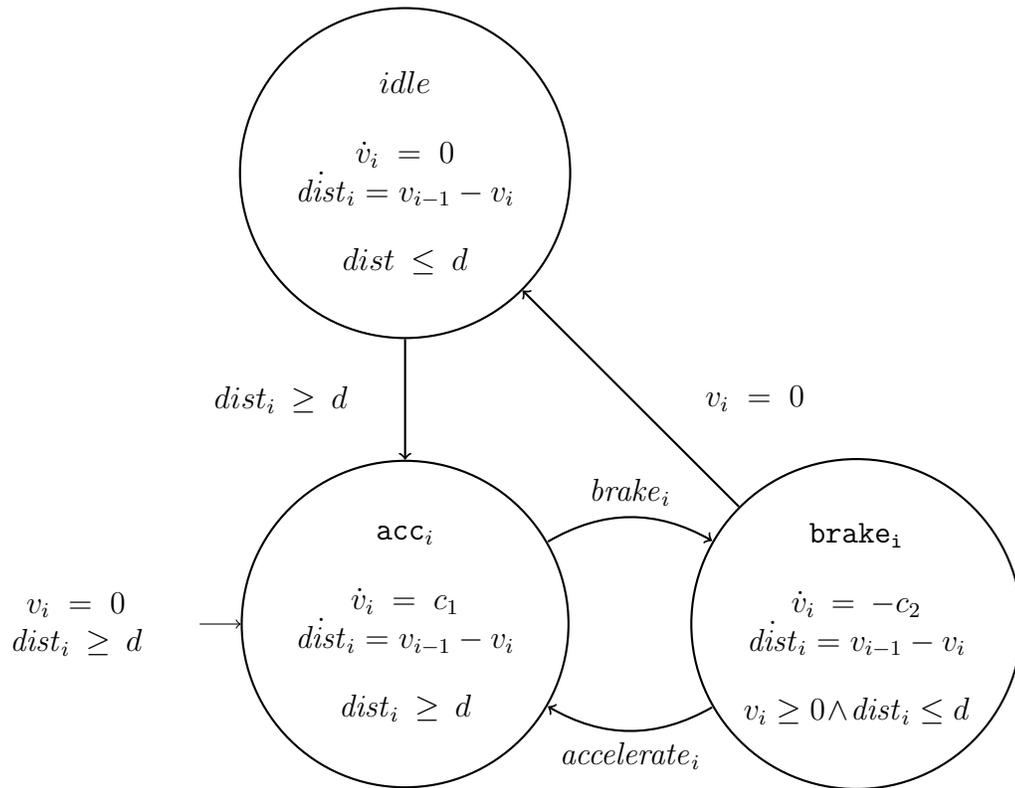
Exercise 2

- a) Give a *hybrid automaton* model for the i th car ($i > 1$) in the following highway car platoon:

- The car i has velocity v_i .
- If the i th car's distance $dist_i$ to the $(i-1)$ st car ahead is at least $d > 0$ then the car accelerates ($accelerate_i, \mathbf{acc}_i$) constantly with an acceleration $\dot{v}_i = c_1$, $c_1 > 0$, as long as $dist_i$ remains at least d .
- If the i th car's distance $dist_i$ to the $(i-1)$ st car ahead is at most d then the car brakes ($brake_i, \mathbf{brake}_i$) constantly with negative acceleration $\dot{v}_i = -c_2$, $c_2 > 0$, as long as its velocity is non-negative and $dist_i$ remains at most d .

Initially our car is standing ($v_i = 0$) $dist_i \geq d$ meters away from its predecessor and is in the accelerating mode. The velocity of the predecessor car is v_{i-1} .

Solution:



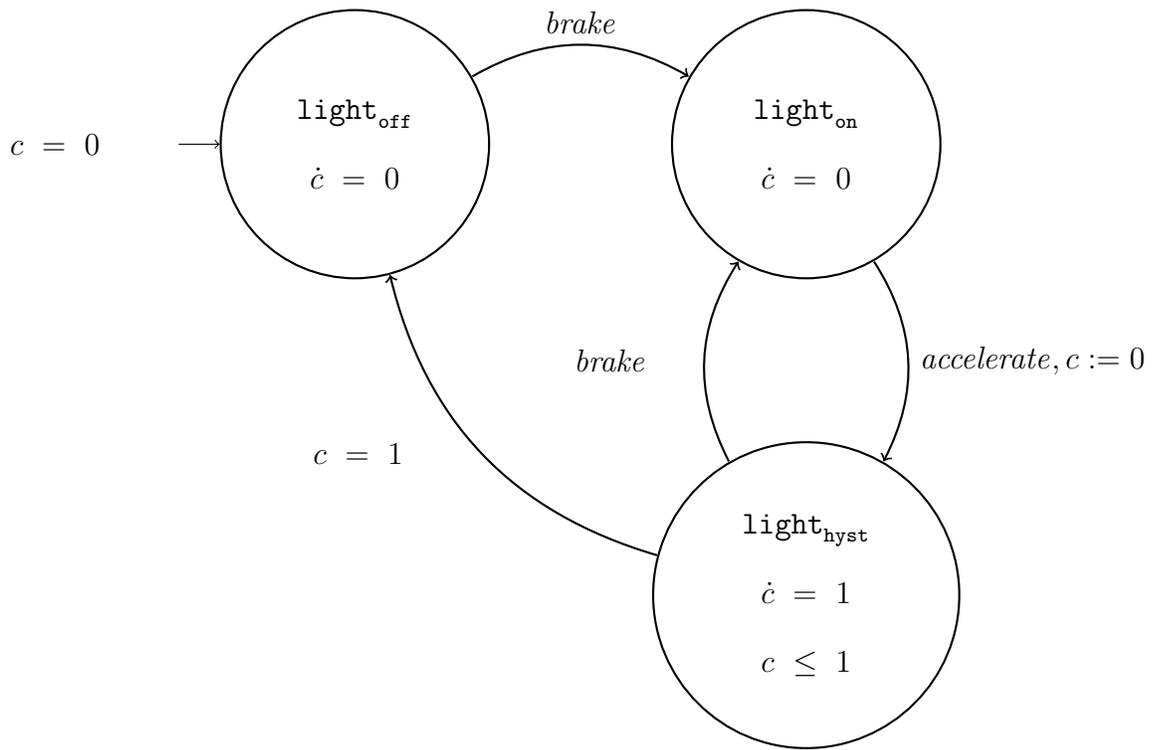
b) What happens, when eventually $dist_i = d$ holds?

Solution:

The car i can constantly switch between its two locations infinitely often in finite (even zero) time. This behavior is called Zeno-behavior.

c) For safety reasons, the car company has decided to introduce a controller for a breaking light, which is (of course) synchronized to the car's behavior and enables the breaking light one second longer (hysteresis) than the braking process of the car. Give a hybrid automaton model for this controller, which makes use of the synchronization labels *accelerate* and *brake* from the previous tasks and where the location labels reflect the state of the breaking light (i.e. $light_{on}$, $light_{hyst}$, $light_{off}$).

Solution:



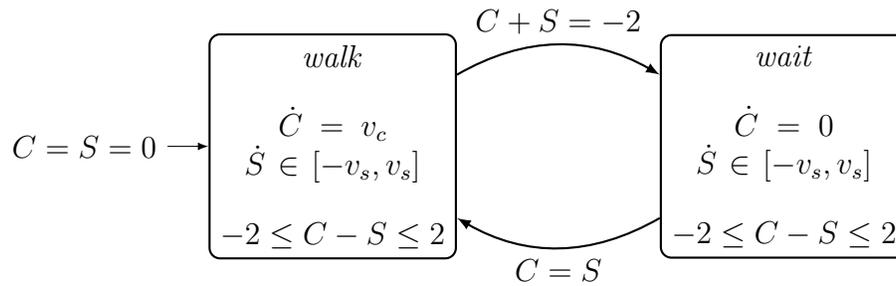
Exercise 3

Charlie Brown walks his dog Snoopy every day the same way:

- Both leave the house next to each other and start their walk.
- As Charlie (C) is thinking about important things (the girl with the red hair), he walks with continuous pace v_c .
- Curious Snoopy (S) is less steady and thus changes his pace between $-v_s$ and v_s , while $0 < v_c < v_s$ holds.
- The leach has only a length of 2 meters. Whenever Snoopy is left behind 2 meters, Charlie waits until Snoopy closes up to him and both continue the walk.

Please give a linear hybrid automaton of type I, which models the position S and C of Snoopy and Charlie respectively.

Solution: The linear hybrid automaton which models the position of Charlie and Snoopy can be specified as follows:

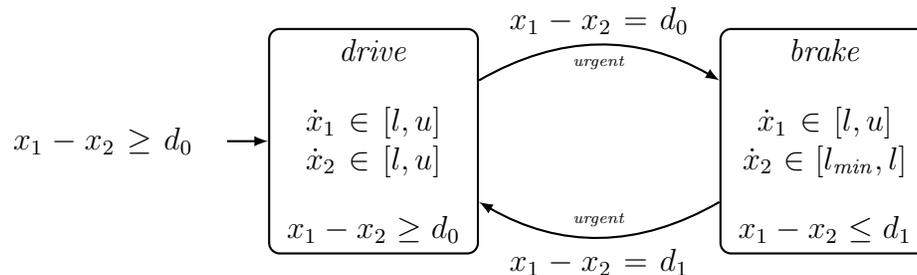


Exercise 4

We consider a vehicle platoon, where two cars are driving with speeds $\dot{x}_i \in [l, u], i \in \{1, 2\}, 0 < l < u$ on a road, such that the 1st car is in front of the 2nd car. The goal is to keep the distance between two cars above some constant $d_0 > 0$. When the distance is at its boundary d_0 , the rear car brakes, which limits its speed to the interval $\dot{x}_2 \in [l_{min}, l], 0 < l_{min} < l$. Additionally we utilize a second constant $d_1 > d_0 > 0$ to prolong the braking process until this target distance d_1 is reached. Initially the goal condition is satisfied.

Note that both transitions are *urgent transitions*, which means that they are taken as soon as they are enabled.

A linear hybrid automaton I model of the above system is given as follows:



a) Please calculate the forward time closure as presented in the lecture.

Solution:

a) We use the algorithm presented in the lecture, where $e_1 = (l_0, x_1^{pre} - x_2^{pre} = d_0, x =$

$x^{pre} \wedge y = y^{pre}, l_1)$ and $e_2 = (l_1, x_1^{pre} - x_2^{pre} = d_1, x = x^{pre} \wedge y = y^{pre}, l_0)$:

$$P_0 = \{(l_0, R_{l_0}^0), (l_1, R_{l_1}^0)\}$$

$$R_{l_0}^0 = T_{l_0}^+(\varphi_{Init} \wedge \varphi_{Inv})$$

$$= T_{l_0}^+(x_1 - x_2 \geq d_0)$$

$$= \exists t. \exists x_1^{pre}. \exists x_2^{pre}. x_1^{pre} - x_2^{pre} \geq d_0 \wedge t \geq 0 \wedge$$

$$x_1 \leq x_1^{pre} + u \cdot t \wedge x_1 \geq x_1^{pre} + l \cdot t \wedge$$

$$x_2 \leq x_2^{pre} + u \cdot t \wedge x_2 \geq x_2^{pre} + l \cdot t \wedge$$

$$x_1 - x_2 \geq d_0$$

$$\stackrel{(F.M.)}{=} \exists t. \exists x_1^{pre}. t \geq 0 \wedge$$

$$x_1 \leq x_1^{pre} + u \cdot t \wedge x_1 \geq x_1^{pre} + l \cdot t \wedge$$

$$x_2 - u \cdot t \leq x_2 - l \cdot t \wedge x_2 - u \cdot t \leq x_1^{pre} - d_0 \wedge$$

$$x_1 - x_2 \geq d_0$$

$$\stackrel{(F.M.)}{=} \exists t. t \geq 0 \wedge$$

$$x_2 - u \cdot t + d_0 \leq x_1 - l \cdot t \wedge$$

$$x_1 - u \cdot t \leq x_1 - l \cdot t \wedge$$

$$x_2 - u \cdot t \leq x_2 - l \cdot t \wedge$$

$$x_1 - x_2 \geq d_0$$

$$\stackrel{(F.M.)}{=} x_1 - x_2 \geq d_0$$

$$R_{l_1}^0 = T_{l_1}^+(\varphi_{Init} \wedge \varphi_{Inv}) = false$$

$$P_1 = \{(l_0, R_{l_0}^1), (l_1, R_{l_1}^1)\}$$

$$R_{l_1}^1 = T_{l_1}^+(D_{e_1}^+(R_{l_0}^0))$$

$$R_{l_1}^1 = T_{l_1}^+(\exists x_1^{pre}. \exists x_2^{pre}.$$

$$x_1^{pre} - x_2^{pre} \geq d_0 \wedge x_1^{pre} - x_2^{pre} = d_0 \wedge x_1 = x_1^{pre} \wedge x_2 = x_2^{pre} \wedge x_1^{pre} - x_2^{pre} \leq d_1)$$

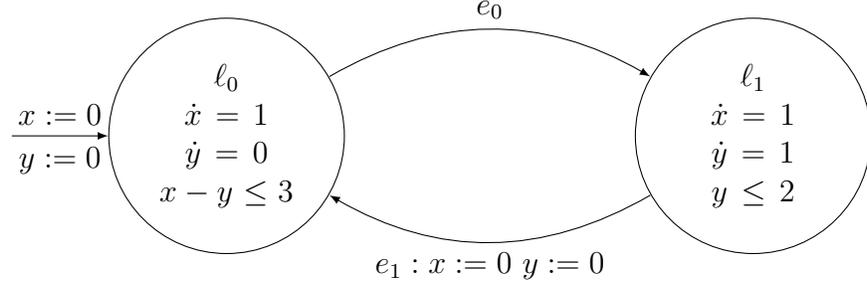
$$= T_{l_1}^+(x_1 - x_2 = d_0 \wedge x_1 - x_2 \leq d_1)$$

$$\begin{aligned}
R_{l_1}^1 &= \exists t. \exists x_1^{pre}. \exists x_2^{pre}. t \geq 0 \wedge \\
&\quad \underline{x_2^{pre} = x_1^{pre} - d_0} \wedge \\
&\quad x_1 \leq x_1^{pre} + u \cdot t \wedge x_1 \geq x_1^{pre} + l \cdot t \wedge \\
&\quad x_2 \leq x_2^{pre} + l \cdot t \wedge x_2 \geq x_2^{pre} + l_{min} \cdot t \wedge \\
&\quad x_1 - x_2 \leq d_1 \\
&= \exists t. \exists x_1^{pre}. t \geq 0 \wedge \\
&\quad x_1 \leq x_1^{pre} + u \cdot t \wedge x_1 \geq x_1^{pre} + l \cdot t \wedge \\
&\quad x_2 \leq x_1^{pre} - d_0 + l \cdot t \wedge x_2 \geq x_1^{pre} - d_0 + l_{min} \cdot t \wedge \\
&\quad x_1 - x_2 \leq d_1 \\
&= \exists t. \exists x_1^{pre}. t \geq 0 \wedge \\
&\quad x_1 - u \cdot t \leq x_1^{pre} \wedge x_1^{pre} \leq x_1 + l \cdot t \wedge \\
&\quad x_2 + d_0 - l \cdot t \leq x_1^{pre} \wedge x_1^{pre} \leq x_2 + d_0 - l_{min} \cdot t \wedge \\
&\quad x_1 - x_2 \leq d_1 \\
&\stackrel{(F.M.)}{=} \exists t. t \geq 0 \wedge \\
&\quad x_1 - u \cdot t \leq x_1 - l \cdot t \wedge x_1 - u \cdot t \leq x_2 + d_0 - l_{min} \cdot t \\
&\quad \wedge x_2 + d_0 - l \cdot t \leq x_1 - l \cdot t \wedge x_2 + d_0 - l_{min} \cdot t \leq x_2 + d_0 - l_{min} \cdot t \\
&\quad \wedge x_1 - x_2 \leq d_1 \\
&= \exists t. t \geq 0 \wedge \\
&\quad \frac{x_1 - x_2 - d_0}{u - l_{min}} \leq t \wedge x_1 - x_2 \geq d_0 \wedge x_1 - x_2 \leq d_1 \\
&\stackrel{(F.M.)}{=} x_1 - x_2 \geq d_0 \wedge x_1 - x_2 \leq d_1 \\
R_{l_0}^2 &= T_{l_0}^+(D_{e_2}^+(R_1^{l_1})) \\
&= T_{l_0}^+(\exists x_1^{pre}. \exists x_2^{pre}. \wedge \\
&\quad d_0 \leq x_1 - x_2 \wedge x_1 - x_2 \leq d_1 \wedge \\
&\quad \underline{x_1^{pre} - x_2^{pre} = d_1} \wedge \underline{x_1 = x_1^{pre}} \wedge \underline{x_2 = x_2^{pre}} \wedge x_1 - x_2 \geq d_0) \\
&= T_{l_1}^+(d_0 \leq x_1 - x_2 \wedge x_1 - x_2 \leq d_1 \wedge x_1 - x_2 = d_1) \\
&= T_{l_1}^+(x_1 - x_2 = d_1) \subseteq R_{l_0}^0
\end{aligned}$$

Reachable set: $R_0^{l_0} \cup R_1^{l_1}$

Exercise 5

Consider the following linear hybrid automaton I:



We define P_i as follows:

$$\begin{aligned} P_0 &= \{R_0\} \\ R_0 &= (\ell_0, x = y = 0) \\ P_i &= D^+(T^+(P_{i-1})) \end{aligned}$$

for $i > 0, l \in \{\ell_0, \ell_1\}, e \in \{e_0, e_1\}$. Please compute P_2 .

Solution:

The set R_i where $i > 0$ is the reachable set of the linear hybrid automaton after i many jumps, and P_0 is the set of initial regions. We compute $P_1 = \{R_1\}$ as follows.

$$\begin{aligned} R_1 &= (\ell_0, D_{e_0}^+(T_{\ell_0}^+(x = 0 \wedge y = 0))) \\ &= (\ell_0, D_{e_0}^+(\exists t. \exists x^{pre}. \exists y^{pre}. t \geq 0 \wedge \underline{x^{pre} = 0 \wedge y^{pre} = 0} \wedge y = y^{pre} \wedge x = x^{pre} + t \wedge x - y \leq 3)) \\ &= (\ell_0, D_{e_0}^+(\exists t. t \geq 0 \wedge y = 0 \wedge \underline{x = t \wedge x \leq 3})) \\ &= (\ell_0, D_{e_0}^+(x \geq 0 \wedge y = 0 \wedge x \leq 3)) \\ &= (\ell_1, x \geq 0 \wedge y = 0 \wedge x \leq 3 \wedge y \leq 2) \\ &= (\ell_1, x \geq 0 \wedge y = 0 \wedge x \leq 3) \end{aligned}$$

The set $P_2 = \{R_2\}$ can be computed similarly.

$$\begin{aligned} R_2 &= (\ell_1, D_{e_1}^+(T_{\ell_1}^+(x \geq 0 \wedge y = 0 \wedge x \leq 3))) \\ &= (\ell_1, D_{e_1}^+(\exists t. \exists x^{pre}. \exists y^{pre}. t \geq 0 \wedge \underline{x^{pre} \geq 0 \wedge y^{pre} = 0 \wedge x^{pre} \leq 3 \wedge x = x^{pre} + t \wedge y = y^{pre} + t \wedge y \leq 2})) \\ &= (\ell_1, D_{e_1}^+(\exists t. \exists x^{pre}. t \geq 0 \wedge \underline{x^{pre} \geq 0 \wedge x^{pre} \leq 3 \wedge x = x^{pre} + t \wedge y = t \wedge y \leq 2})) \\ &= (\ell_1, D_{e_1}^+(\exists x^{pre}. \underline{y \geq 0 \wedge x^{pre} \geq 0 \wedge x^{pre} \leq 3 \wedge x = x^{pre} + y \wedge y \leq 2})) \\ &= (\ell_1, D_{e_1}^+(y \geq 0 \wedge x - y \geq 0 \wedge x - y \leq 3 \wedge y \leq 2)) \\ &= (\ell_0, x = 0 \wedge y = 0) \end{aligned}$$

b) Please sketch the reachable area in the map.

Solution:

a) We apply forward analysis, as presented in the lecture, using the following notations:

$$\begin{aligned}
l_0 &:= \textit{island_stream} \\
l_1 &:= \textit{coastline_stream} \\
\textit{Init} &:= \{(l_0, \underbrace{x = 0 \wedge y = 0}_{=: \textit{Init}_{l_0}})\} \\
\textit{Inv} &:= \{(l_0, \underbrace{0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50}_{=: \textit{Inv}_{l_0}}, \\
&\quad (l_1, \underbrace{50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100}_{=: \textit{Inv}_{l_1}})\} \\
\textit{Act} &:= \{(l_0, \underbrace{x + 4.5t \leq x' \wedge x' \leq x + 6t \wedge y' = y + 3t}_{=: \textit{Act}_{l_0}}, \\
&\quad (l_1, \underbrace{x - 10t \leq x' \wedge x' \leq x - 4t \wedge y' = y + 10t}_{=: \textit{Act}_{l_1}})\} \\
P^{bad} &:= \{(l_1, \underbrace{80 \leq x \wedge x \leq 100 \wedge 80 \leq y \wedge y \leq 100}_{=: \varphi^{bad}})\}
\end{aligned}$$

Furthermore we use e to denote the only transition in the system:

$$e := (l_0, \underbrace{50 \leq x \wedge x \leq 100 \wedge y = 50}_{=: \varphi_e^{guard}}, \underbrace{x' = x \wedge y' = y}_{=: \varphi_e^{reset}}, l_1).$$

$$\begin{aligned}
P^0 &= T^+(\text{Init} \hat{\wedge} \text{Inv}) \\
&= T^+(\{(l_0, \underbrace{x = 0 \wedge y = 0}_{\text{Init}_{l_0}} \wedge \underbrace{0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50}_{\text{Inv}_{l_0}})\}) \\
&= T^+(\{(l_0, \underbrace{x = 0 \wedge y = 0}_{=:\varphi_{l_0}^0})\}) \\
&= \{(l_0, \underbrace{T_{l_0}^+(x = 0 \wedge y = 0)}_{\varphi_{l_0}^0})\} \\
&= \{(l_0, \exists t. \boxed{\exists x^{pre}}. \boxed{\exists y^{pre}}. 0 \leq t \wedge \underbrace{x^{pre} = 0 \wedge y^{pre} = 0}_{\varphi_{l_0}^0[x^{pre}, y^{pre}/x, y]} \wedge \\
&\quad \underbrace{x^{pre} + 4.5t \leq x \wedge x \leq x^{pre} + 6t \wedge y = y^{pre} + 3t}_{\text{Act}_{l_0}[x^{pre}, y^{pre}, x, y/x, y, x', y']} \wedge \\
&\quad \underbrace{0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50}_{\text{Inv}_{l_0}})\} \\
&\stackrel{\text{elim. } x^{pre}, y^{pre}}{=} \{(l_0, \boxed{\exists t}. 0 \leq t \wedge 4.5t \leq x \wedge x \leq 6t \wedge \boxed{y = 3t} \wedge 0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50)\} \\
&\stackrel{\text{elim. } t}{=} \{(l_0, 0 \leq \frac{1}{3}y \wedge 4.5 \cdot \frac{1}{3}y \leq x \wedge x \leq 6 \cdot \frac{1}{3}y \wedge 0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50)\} \\
&= \{(l_0, 1.5y \leq x \wedge x \leq 2y \wedge 0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50)\} \\
P^0 \hat{\wedge} P^{bad} &= \emptyset
\end{aligned}$$

$$R := (l_0, \underbrace{1.5y \leq x \wedge x \leq 2y \wedge 0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50}_{=:\varphi^R})$$

$$R' := (l_1, D_e^+(\varphi^R))$$

$$\begin{aligned}
&= (l_1, \boxed{\exists x^{pre}}. \boxed{\exists y^{pre}}. \\
&\quad \underbrace{1.5y^{pre} \leq x^{pre} \wedge x^{pre} \leq 2y^{pre} \wedge 0 \leq x^{pre} \wedge x^{pre} \leq 100 \wedge 0 \leq y^{pre} \wedge y^{pre} \leq 50}_{\varphi^R[x^{pre}, y^{pre}/x, y]} \wedge
\end{aligned}$$

$$50 \leq x^{pre} \wedge x^{pre} \leq 100 \wedge y^{pre} = 50 \wedge$$

$$\varphi_e^{guard}[x^{pre}, y^{pre}/x, y]$$

$$\boxed{x = x^{pre}} \wedge \boxed{y = y^{pre}} \wedge$$

$$\varphi_e^{reset}[x^{pre}, y^{pre}, x, y/x, y, x', y']$$

$$50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100)$$

$$\text{Inv}_{l_1}$$

$$\begin{aligned}
&\stackrel{\text{elim. } x^{pre}, y^{pre}}{=} (l_1, \underbrace{75 \leq x \wedge x \leq 100 \wedge y = 50}_{=:\varphi^{R'}})
\end{aligned}$$

$$\begin{aligned}
R'' &:= (l_1, T_{l_1}^+(\varphi^{R'})) \\
&= (l_1, \exists t. \exists x^{pre}. \boxed{\exists y^{pre}}. \underbrace{0 \leq t \wedge 75 \leq x^{pre} \wedge x^{pre} \leq 100 \wedge \boxed{y^{pre} = 50}}_{\varphi^{R'}[x^{pre}, y^{pre}/x, y]} \wedge \\
&\quad \underbrace{x^{pre} - 10t \leq x \wedge x \leq x^{pre} - 4t \wedge y = y^{pre} + 10t}_{\text{Act}_{l_1}} \wedge \\
&\quad \underbrace{50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100}_{\text{Inv}_{l_1}}) \\
&\stackrel{\text{elim. } y^{pre}}{=} (l_1, \exists t. \exists x^{pre}. 0 \leq t \wedge 75 \leq x^{pre} \wedge x^{pre} \leq 100 \wedge \\
&\quad x^{pre} - 10t \leq x \wedge x \leq x^{pre} - 4t \wedge y = 50 + 10t \wedge \\
&\quad 50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100) \\
&\stackrel{\text{elim. } x^{pre}}{=} (l_1, \boxed{\exists t}. 0 \leq t \wedge 75 \leq x + 10t \wedge x + 4t \leq 100 \wedge \\
&\quad \boxed{y = 50 + 10t} \wedge \\
&\quad 50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100) \\
&\stackrel{\text{elim. } t}{=} (l_1, \underbrace{125 \leq x + y \wedge x + 0.4y \leq 120 \wedge 50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100}_{=: \varphi^{R''}})
\end{aligned}$$

$\{R''\} \hat{\cap} P^{bad} \neq \emptyset$ iff $\exists x. \exists y.$

$$\begin{aligned}
&\quad \underbrace{125 \leq x + y \wedge x + 0.4y \leq 120 \wedge 50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100}_{\varphi^{R''}} \wedge \\
&\quad \underbrace{80 \leq x \wedge x \leq 100 \wedge 80 \leq y \wedge y \leq 100}_{\varphi^{bad}} \\
&= \boxed{\exists x}. \exists y. \overline{125 \leq x + y \wedge x + 0.4y \leq 120} \wedge \\
&\quad \overline{80 \leq x \wedge x \leq 100} \wedge 80 \leq y \wedge y \leq 100 \\
&\stackrel{\text{elim. } x}{=} \exists y. 25 \leq y \wedge y \leq 100 \wedge 8\frac{1}{3} \leq y \wedge 80 \leq y \wedge y \leq 100 \\
&= \boxed{\exists y}. \overline{80 \leq y \wedge y \leq 100} \\
&\stackrel{\text{elim. } y}{=} \text{true} \rightarrow \{R_{l_1}^1\} \hat{\cap} P^{bad} \neq \emptyset
\end{aligned}$$

Thus there exists a possibility of a rescue.

b) We can use the previously computed constraints to visualize the reachable state space:

