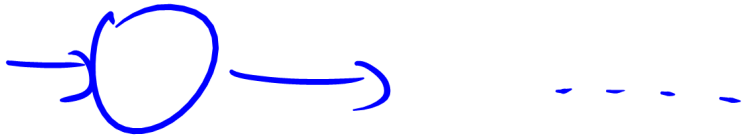
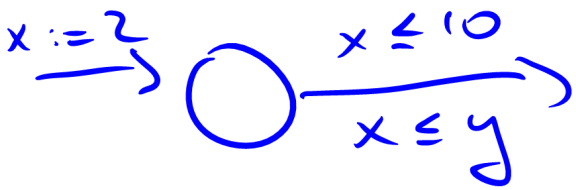
LSTS:

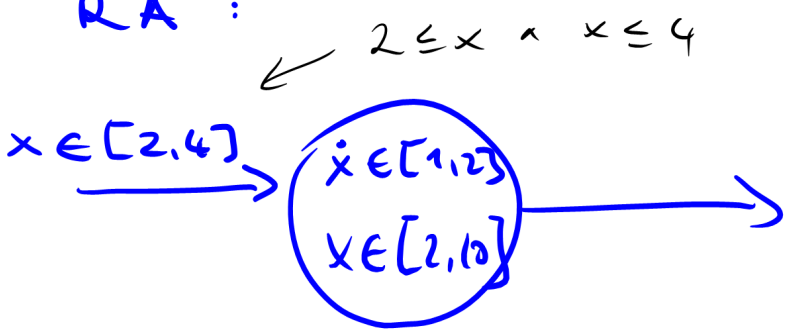


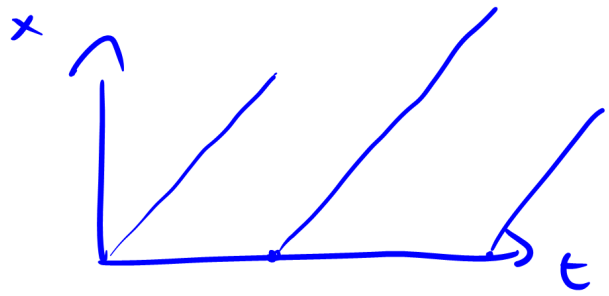
LTS:

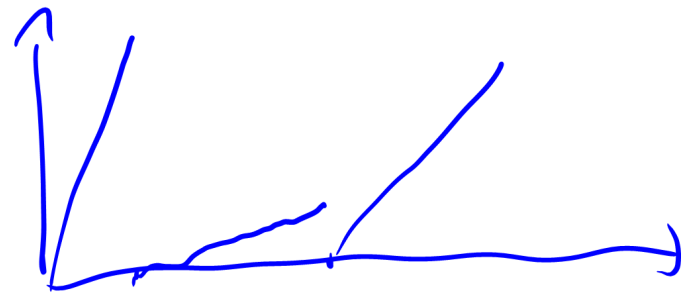


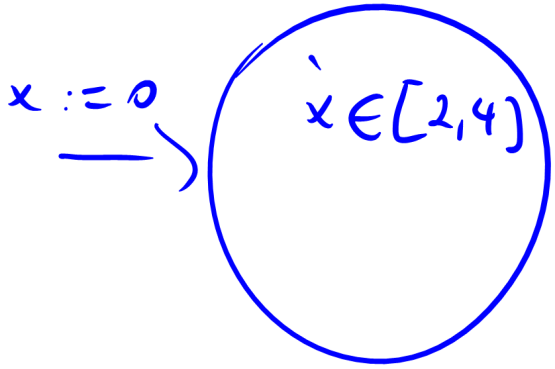
TA:

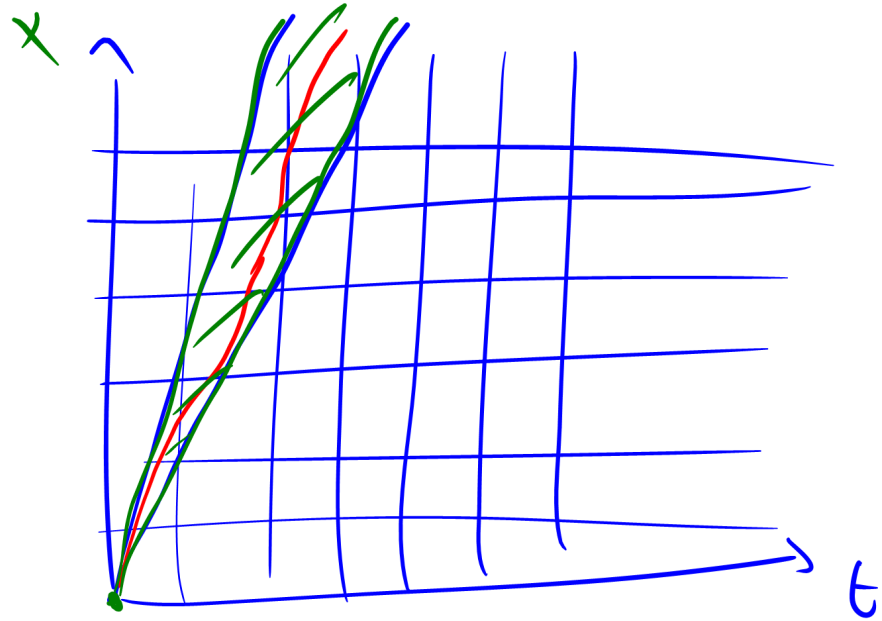
RA:

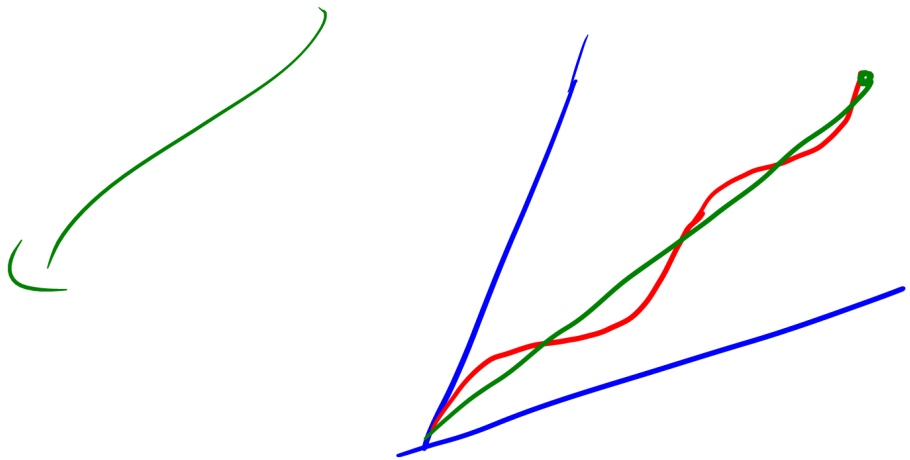


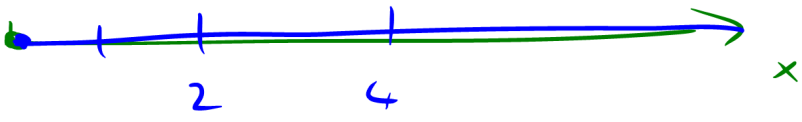





x := 0
ẋ ∈ [2,4]


x
t




2
4
x

Modeling and Analysis of Hybrid Systems

Linear hybrid automata I

Rectangular automata

Prof. Dr. Erika Ábrahám

Informatik 2 - LuFG Theory of Hybrid Systems
RWTH Aachen University

Szeged, Hungary, 27 September - 06 October 2017

Literature

Henzinger et al.: What's decidable about hybrid automata?

Journal of Computer and System Sciences, 57:94–124, 1998

Motivation

- The special class of timed automata with TCTL is decidable, thus model checking is possible.
- What about more expressive model classes for hybrid systems?

What is decidable about hybrid automata?

Two central problems for the analysis of hybrid automata:

- Safety: The problem to decide whether something "bad" can happend during the execution of a system.
- Liveness: The problem to decide whether there is always the possibility that something "good" will eventually happen during the execution of a system.

Both problems are decidable in certain special cases, and undecidable in certain general cases.

What is decidable about hybrid automata?

A particularly interesting class:

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A particularly interesting class:

- all conditions, effects, and flows are described by rectagular sets.

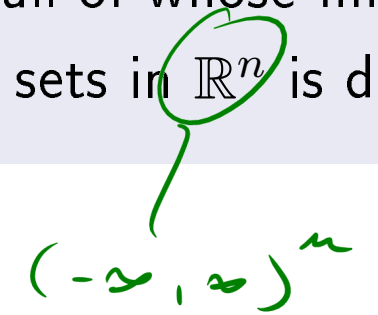
What is decidable about hybrid automata?

A particularly interesting class:

- all conditions, effects, and flows are described by rectagular sets.

Definition

- A set $\mathcal{R} \subset \mathbb{R}^n$ is rectangular if it is a cartesian product of (possibly unbounded) intervals, all of whose finite endpoints are rationals.
- The set of rectangular sets in $\mathbb{R}^n$ is denoted $\mathcal{R}^n$.



Rectangular automaton

Definition

A rectangular automaton A is a tuple $\mathcal{H} = (Loc, Var, Lab, Edge, Act, Inv, Init)$ with

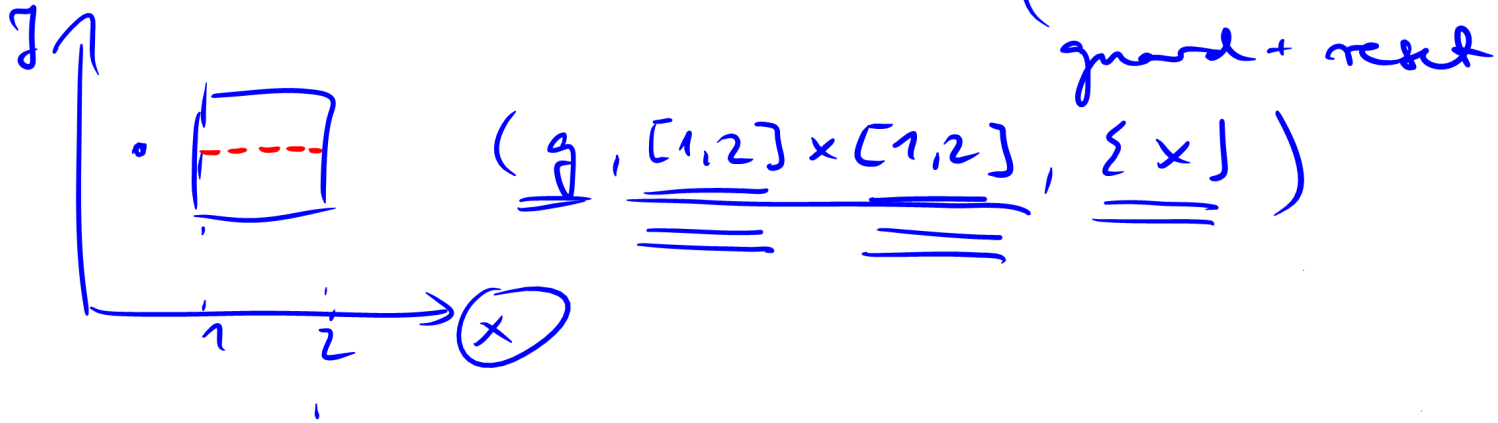
- finite set of locations Loc,
- finite set of real-valued variables $Var = \{x_1, \ldots, x_n\}$,
- finite set of synchronization labels Lab,
- finite set of edges $Edge \subseteq Loc \times Lab \times \mathcal{R}^n \times \mathcal{R}^n \times 2^{\{1,\ldots,n\}} \times Loc$,
- a flow function $Act : Loc \to \mathcal{R}^n$,
- an invariant function $Inv : Loc \to \mathcal{R}^n$,
- initial states $Init : Loc \to \mathcal{R}^n$.

variables to reset

guard

"reset"

guard + reset



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- States: $\sigma = (l, \vec{x}) \in (Loc \times \mathbb{R}^n)$ with $\vec{x} \in Inv(l)$

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$=$

$\Rightarrow$ g Formula $\rightarrow$ defines

$\Rightarrow$ r rectangle

$r = Sat(g)$

- States: $\sigma = (l, \vec{x}) \in (Loc \times \mathbb{R}^n)$ with $\vec{x} \in Inv(l)$
- State space: $\Sigma \subseteq Loc \times \mathbb{R}^n$ is the set of all states
- Is the state space rectangular?

Rectangular automaton

- Flows: first time derivatives of the flow trajectories in location $l \in Loc$ are within $Act(l)$
- Jumps: $e = (l, a, pre, post, jump, l') \in Edge$ may move control from location l to location l' starting from a valuation in pre, changing the value of each variable x_i to a nondeterministically chosen value from $post_i$ (the projection of $post$ to the ith dimension), such that the values of the variables $x_i \notin jump$ are unchanged.

Operational semantics

$\forall x.\ Act(e)(x) = [a_x, b_x]$ $\quad t \geq 0$ $\quad \nu' \models Inv(\ell)$

$$\frac{\ell = \ell' \quad \forall x \in Var.\ \nu(x) + a_x t \leq \nu'(x) \leq \nu(x) + b_x t}{(\ell, \nu) \xrightarrow{t} (\ell', \nu')} \text{ Rule time}$$

$\nu' \in Inv(\ell')$ $\quad \nu' \in r$ $\quad \forall x \in Var \setminus S.\ \nu(x) = \nu'(x)$

$$\frac{(\ell, a, (g, r, S), \ell') \in Edge \quad \nu \in g}{(\ell, \nu) \xrightarrow{a} (\ell', \nu')} \text{ Rule jump}$$

$g = \{\nu \mid \nu(x) \in [2,4]\}$

$r = \{\nu \mid \nu(x) \in [5,6]\}$ $\Rightarrow$ transition is never enabled

$S = \emptyset$

Operational semantics

$$\frac{(l, a, \mathit{pre}, \mathit{post}, \mathit{jump}, l') \in Edge \qquad \vec{x} \in \mathit{pre} \quad \vec{x}' \in \mathit{post} \quad \forall i \notin \mathit{jump}.\ x'_i = x_i \quad \vec{x}' \in Inv(l')}{(l, \vec{x}) \xrightarrow{a} (l', \vec{x}')}\ \texttt{Rule}_{\texttt{Discrete}}$$

Operational semantics

$$\frac{(l, a, \mathit{pre}, \mathit{post}, \mathit{jump}, l') \in Edge \qquad \vec{x} \in \mathit{pre} \quad \vec{x}' \in \mathit{post} \quad \forall i \notin \mathit{jump}.\ x'_i = x_i \quad \vec{x}' \in Inv(l')}{(l, \vec{x}) \xrightarrow{a} (l', \vec{x}')} \text{ Rule } \texttt{Discrete}$$

$$\frac{(t = 0 \wedge \vec{x} = \vec{x}') \vee (t > 0 \wedge (\vec{x}' - \vec{x})/t \in Act(l)) \quad \vec{x}' \in Inv(l)}{(l, \vec{x}) \xrightarrow{t} (l, \vec{x}')} \text{ Rule } \texttt{Time}$$

Operational semantics

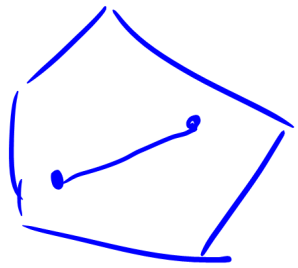
$$\frac{(l, a, \mathit{pre}, \mathit{post}, \mathit{jump}, l') \in \mathit{Edge} \qquad \vec{x} \in \mathit{pre} \quad \vec{x}' \in \mathit{post} \quad \forall i \notin \mathit{jump}.\ x'_i = x_i \quad \vec{x}' \in \mathit{Inv}(l')}{(l, \vec{x}) \xrightarrow{a} (l', \vec{x}')} \text{ Rule } \texttt{Discrete}$$

$$\frac{(t = 0 \wedge \vec{x} = \vec{x}') \vee (t > 0 \wedge (\vec{x}' - \vec{x})/t \in \mathit{Act}(l)) \quad \vec{x}' \in \mathit{Inv}(l)}{(l, \vec{x}) \xrightarrow{t} (l, \vec{x}')} \text{ Rule } \texttt{Time}$$

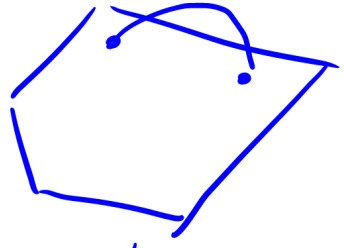
- Execution step: $\rightarrow = \xrightarrow{a} \cup \xrightarrow{t}$
- Path: $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \vec{x}_0)$, $\vec{x}_0 \in \mathit{Inv}(l_0)$
- Initial path: path $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \vec{x}_0)$, $\vec{x}_0 \in \mathit{Init}(l_0)$
- Reachability of a state: exists an initial path leading to the state

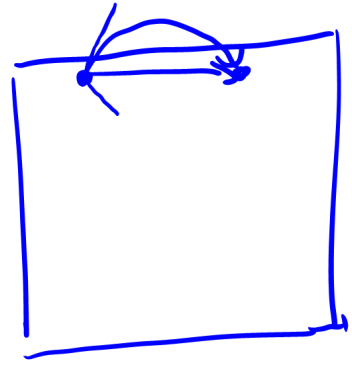
How do we know that the invariant holds all the time during a time step?

① Convex invariants

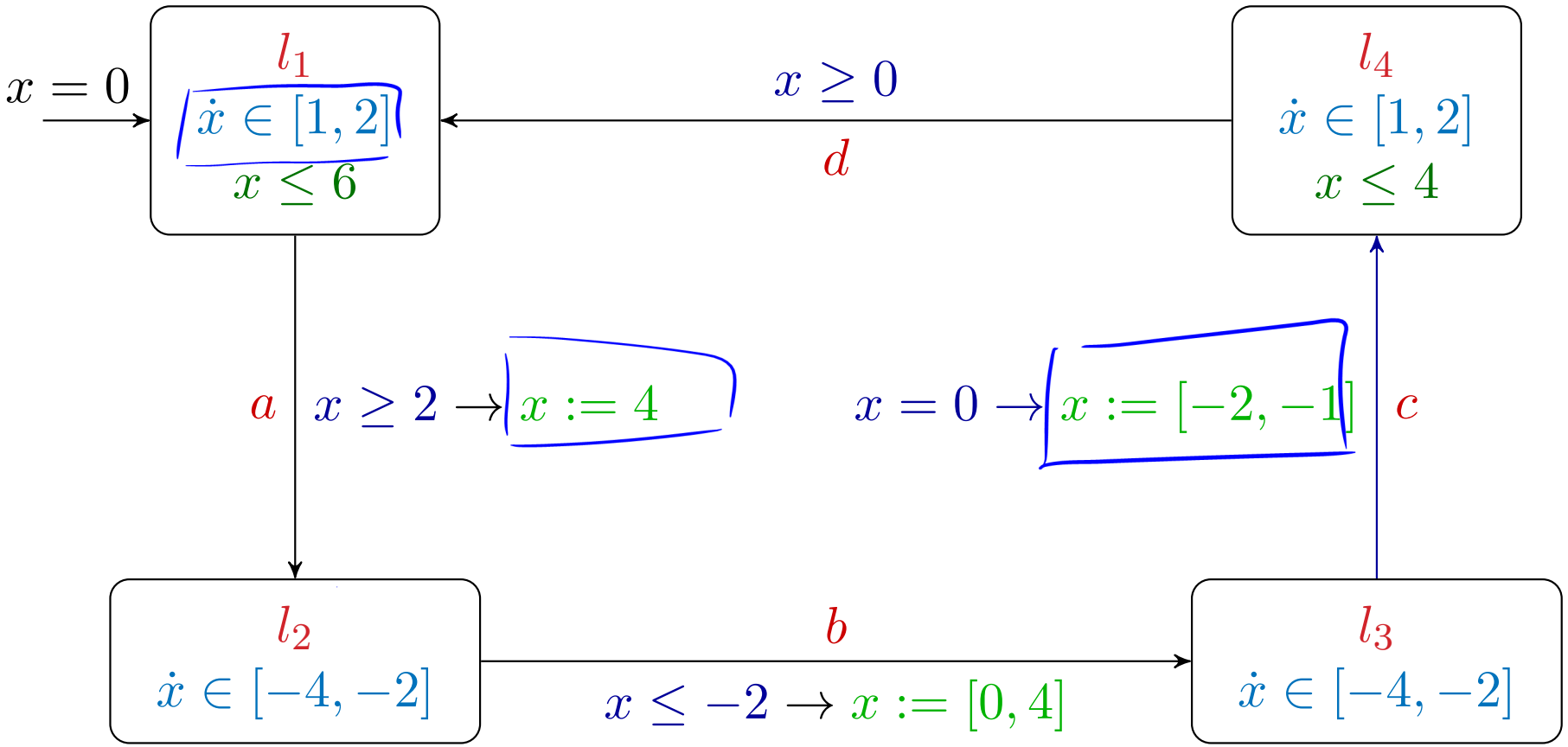


② Nonlinear path violating invariant → exists also linear path not violating the invariant





Example rectangular automaton



Remarks

- If we replace rectangular sets with linear sets, we obtain linear hybrid automata, a super-class of rectangular automata.
- A timed automaton is a special rectangular automaton.

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- If we replace rectangular sets with linear sets, we obtain linear hybrid automata, a super-class of rectangular automata.
- A timed automaton is a special rectangular automaton.

This class lies at the boundary of decidability.

Decidability

The reachability problem is decidable for initialized rectangular automata:

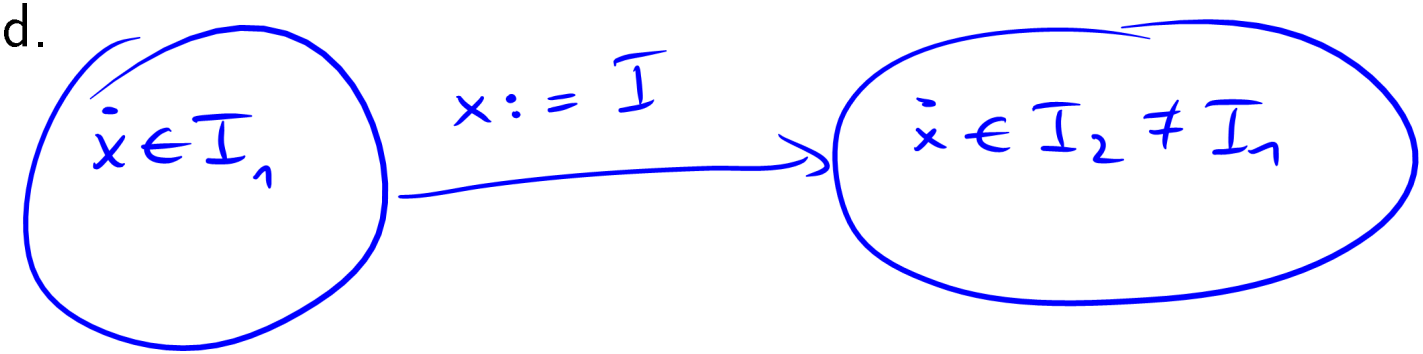
Decidability

The reachability problem is decidable for initialized rectangular automata:

Definition

A rectangular automaton A is initialized, if for every edge $(l, a, pre, post, jump, l')$ of A, and every variable index $i \in \{1, \ldots, n\}$ with $Act(l)_i \neq Act(l')_i$, we have that $i \in jump$.

The reachability problem becomes undecidable if one of the restrictions is relaxed.



makes the problem undecidable

$\exists\, i \in \mathbb{N}_0.$ ~~$\notin$~~

$\sigma_i = (l_i, v_i)$

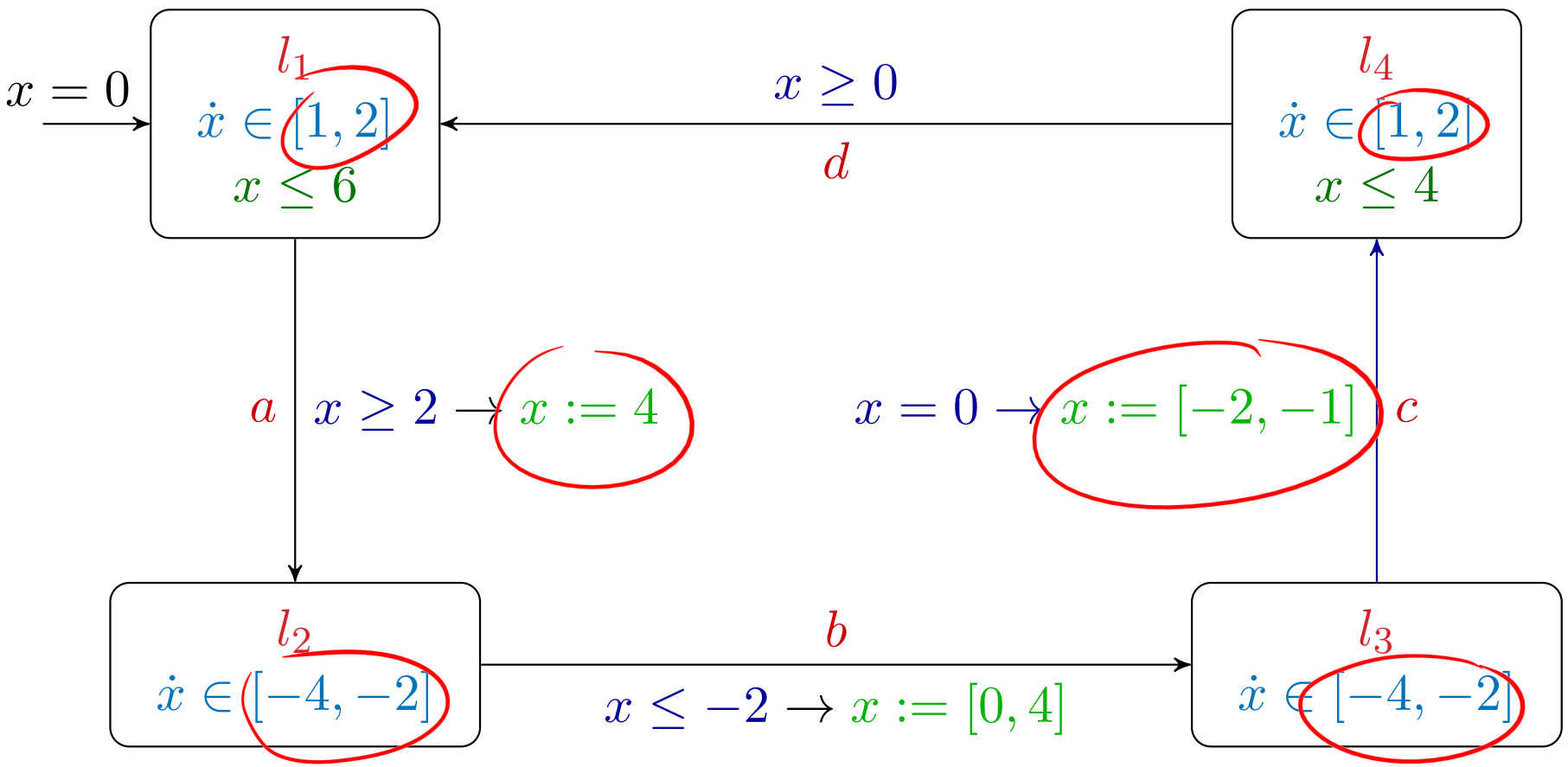
$\dot{x} = c_1$
$\vdots$
$\dot{x} = c_n$

$\sigma_0 \to \cdots \to \sigma_i$

conjunction linear real arithmetic

↓ polynomial complexity

Initialized rectangular automaton



This rectangular automaton is initialized.

What we already know

A timed automaton is a special rectangular automaton such that

- $Init(l)$ is empty or a singleton for each $l \in Loc$,
- for each edge, $post_i$ is a single value for each $i \in jump$ and
- every variable is a clock, i.e., $Act(l)(x) = [1, 1]$ for all locations l and variables x.

Note: here we allow initialization and reset of clocks to any values.

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What we know:

Lemma

The reachability problem for timed automata is complete for PSPACE.

Decidability results

Lemma

The reachability problem for initialized rectangular automata is complete for PSPACE.

Decidability results

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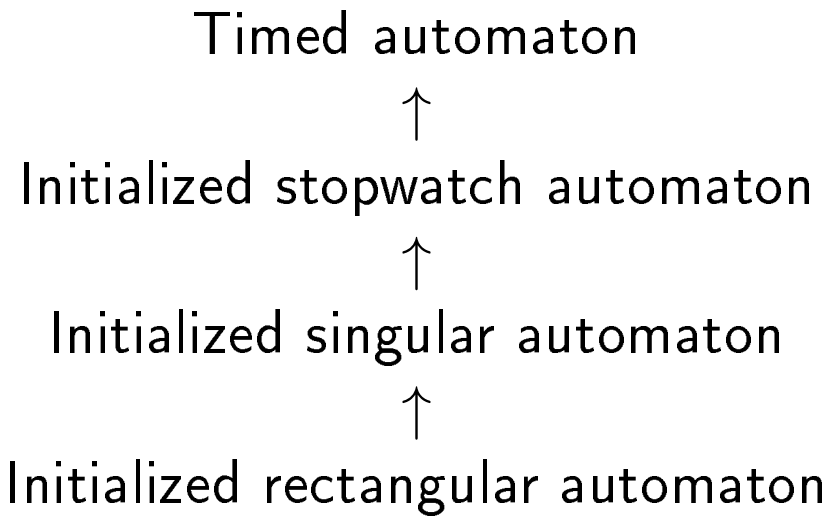


Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

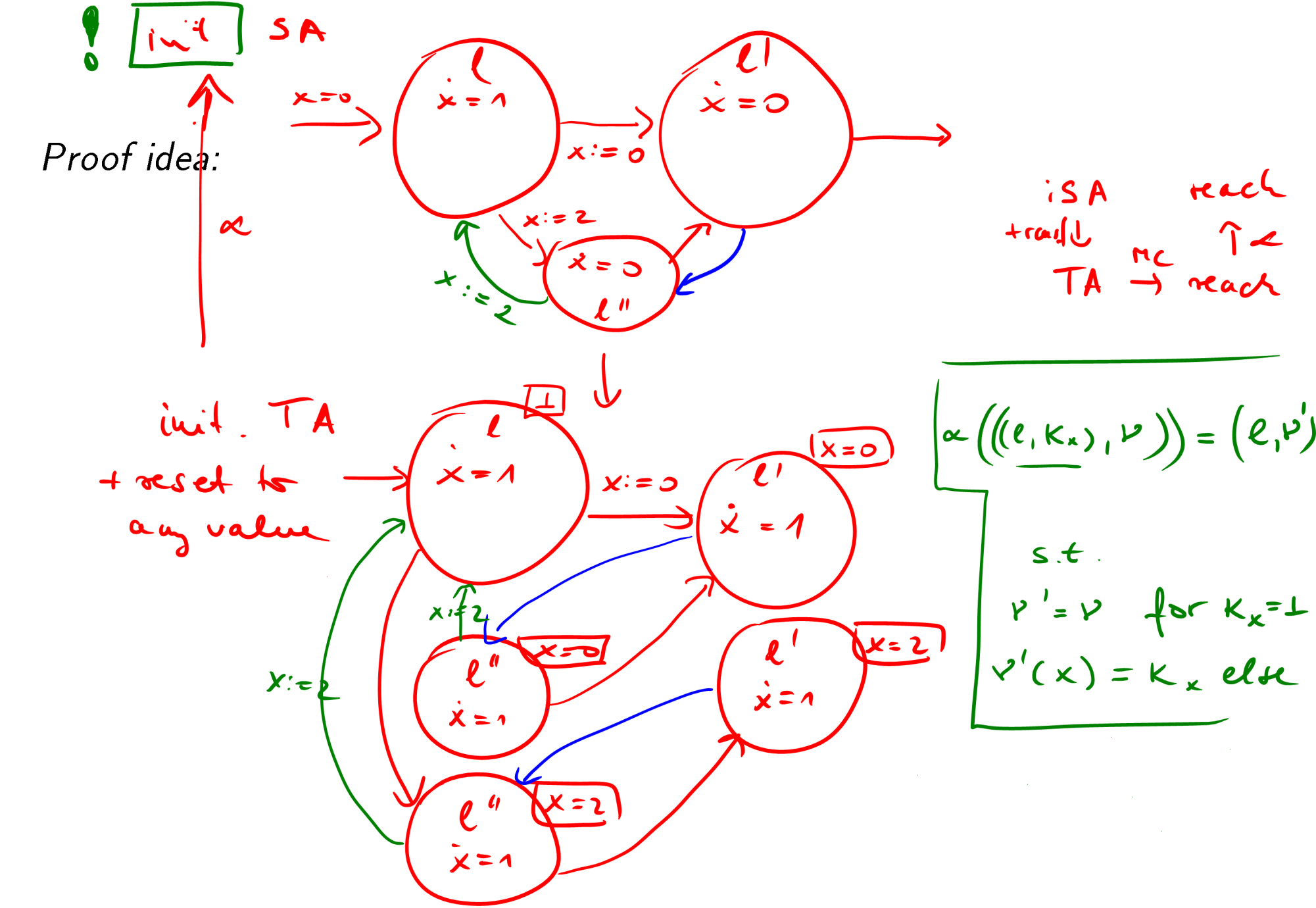
Initialized stopwatch automata

- A stopwatch is a variable with derivatives 0 or 1 only.
- A stopwatch automaton is as a timed automaton but allowing stopwatch variables instead of clocks.
- Initialized stopwatch automata can be polynomially encoded by timed automata.

Lemma

The reachability problem for initialized stopwatch automata is complete for PSPACE.

However, the reachability problem for non-initialized stopwatch automata is undecidable.


init
SA
Proof idea:
α
x=0
ℓ
ẋ = 1
x := 0
ℓ'
ẋ = 0
x := 2
x = 2
ẋ = 0
ℓ''
iSA
reach
+ rail
TA
MC
reach
init. TA
+ reset to
any value
1
x=0
x=0
x=2
x=2
x := 2
x = 2
α((ℓ, K_x), ν) = (ℓ, ν')
s.t.
ν' = ν for K_x = 1
ν'(x) = K_x else

$x \in C(1,4) \iff 1 \leq x \wedge x \leq 4$

$x := c$

Proof idea:

Assume that C is an n-dimensional initialized stopwatch automaton. Let κ be the set of constants used in the definition of C, and let $\kappa_- = \kappa \cup \{-\}$. For each $(k_1, \ldots, k_n) \in \kappa_-^n$, let $\alpha_{k_1,\ldots,k_n} : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\alpha_{k_1,\ldots,k_n}(\vec{x}) = \vec{y}$ such that $y_i = x_i$ if $k_i = -$, and $y_i = k_i$ if $k_i \neq -$.

We define an n-dimensional timed automaton D with locations $Loc_D = Loc_C \times \kappa_-^n$. Each state $\sigma = ((l, k_1, \ldots, k_n), \vec{x})$ of D represents the state $\alpha(\sigma) = (l, \alpha_{k_1,\ldots,k_n}(\vec{x}))$ of C. We extend α to state sets the natural way.

Intuitively, if the ith stopwatch of C is running (slope 1), then its value is tracked by the value of the ith clock of D; if the ith stopwatch is halted (slope 0) at value $k \in \kappa$, then this value is remembered by the current location of D.

Proof idea(continued):

The other components of D are derived from C as follows: $Var_D = Var_C$, $Lab_D = Lab_C$, $Act_D(l) = \Pi_{i=1}^{n}[1,1]$, and $Inv_D(l, k_1, \ldots, k_n) = \alpha_{k_1,\ldots,k_n}(Inv_C(l))$.

For the initial states, assume a location l with stopwatch derivatives $d_1, \ldots, d_n$ (note: each d_i is either 1 or 0). If $Init_C(l)$ is empty then $Init_D(l, \cdot)$ are all empty. Otherwise, $Init(l)$ contains a single value $x_1, \ldots, x_n$. Let k'_i be $-$ if $d_i = 1$ and x_i otherwise. Then $Init_D(l, k'_1, \ldots, k'_n) = Init(l)$ and $Init_D(l, \cdot)$ is empty for all other cases.

Edges: exercise.

We have:

- Each state σ of C is time-abstract bisimilar to the state $\alpha(\sigma)$ of D.
- The reachable set of $Reach(C)$ of C is $\alpha(Reach(D))$.

Decidability results

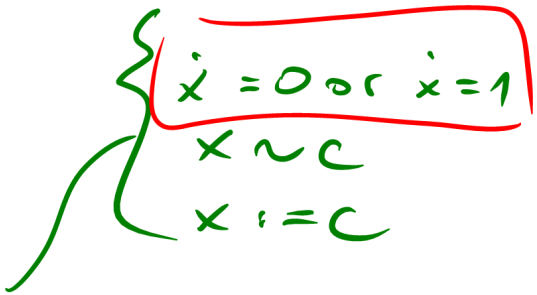
Timed automaton

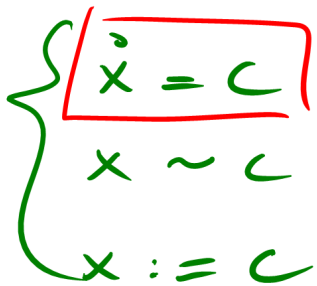
↑

Initialized stopwatch automaton

↑

Initialized singular automaton





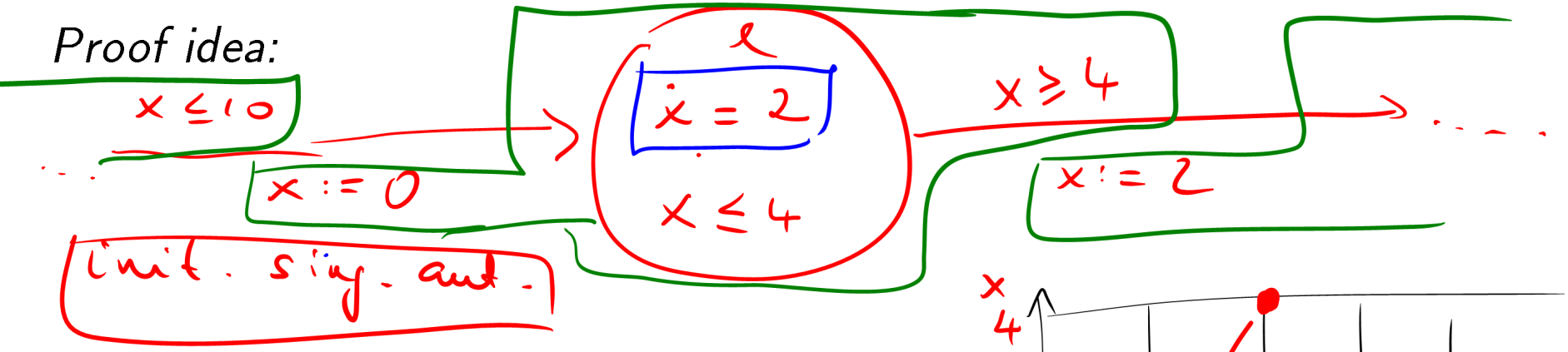
Initialized singular automata

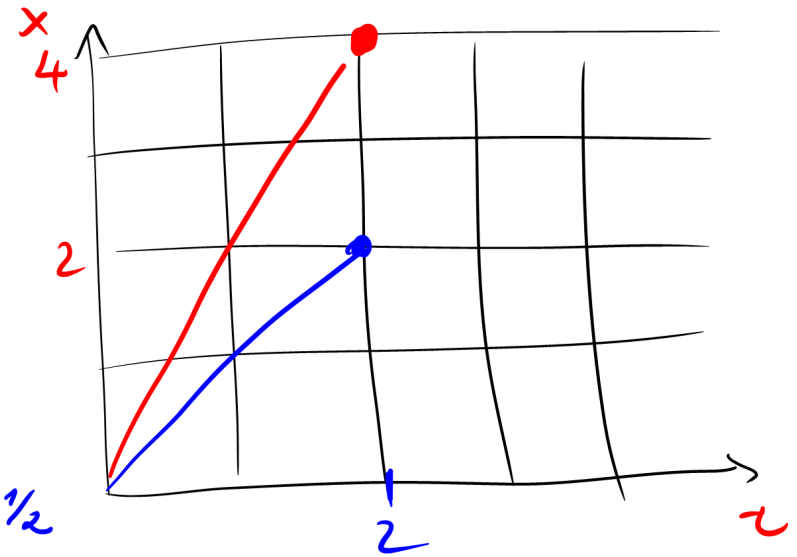
- A variable x_i is a finite-slope variable if $\mathit{flow}(l)_i$ is a singleton in all locations l.
- A singular automaton is as a stopwatch automaton but allowing finite-slope variables instead of stopwatches.
- Initialized singular automata can be polynomially encoded by initialized stopwatch automata.

Lemma

The reachability problem for initialized singular automata is complete for PSPACE.

Proof idea:





$p((\ell, \nu)) = (\ell, \nu')$

$\nu(x) = \nu'(x)$ if $\beta_{\ell,x} \in \{0,1\}$

$\nu'(x) = \nu(x) \cdot 1/\beta_{\ell,x}$

Proof idea: Let B be an n-dimensional initialized singular automaton and let $k_{l,i}$ denote the derivative of the ith variable in location $l \in Loc_B$ of B (i.e., $Act_B(l) = \Pi_{i=1}^{n}[k_{l,i}, k_{l,i}]$).
Let furthermore $\beta_{l,i} = 1/k_{l,i}$ if $k_{l,i} \neq 0$ and $\beta_{l,i} = 1$ otherwise.
For each location $l \in Loc_B$ of B we define a function $\beta_l : \mathbb{R}^n \to \mathbb{R}^n$ by setting $\beta_l(x_1, \ldots, x_n) = (\beta_{l,1} \cdot x_1, \ldots, \beta_{l,n} \cdot x_n)$. β_l can be viewed as a rescaling of the state space, and can be extended naturally to regions ($\beta_l(\Pi_{i=1}^{n}[v_i, v_i']) = \Pi_{i=1}^{n}[\beta_l(v_i), \beta_l(v_i')]$) and states ($\beta((l, x)) = (l, \beta_l(x))$).
We define an n-dimensional initialized stopwatch automaton C, such that all regions in the automaton B occur accordingly rescaled in C. The components of C are the same as B except the invariants $Inv_C(l) = \beta_l(Inv_B(l))$, the activities $Act_C(l) = \beta_l(Act_B(l))$, the initial regions $Init_C(l) = \beta_l(Init_B(l))$, and for each edge $e = (l, a, pre, post, jump, l') \in Edge_B$ in B there is a corresponding edge $e' = (l, a, \beta_l(pre), \beta_{l'}(post), jump, l') \in Edge_C$ in C.
We have:

- Each state σ of B is time-abstract bisimilar to the state $\beta(\sigma)$ of C.
- The reachable set of $Reach(B)$ of B is $\beta(Reach(C))$.

Decidability results

Timed automaton

↑

Initialized stopwatch automaton

↑

Initialized singular automaton

↑

Initialized rectangular automaton

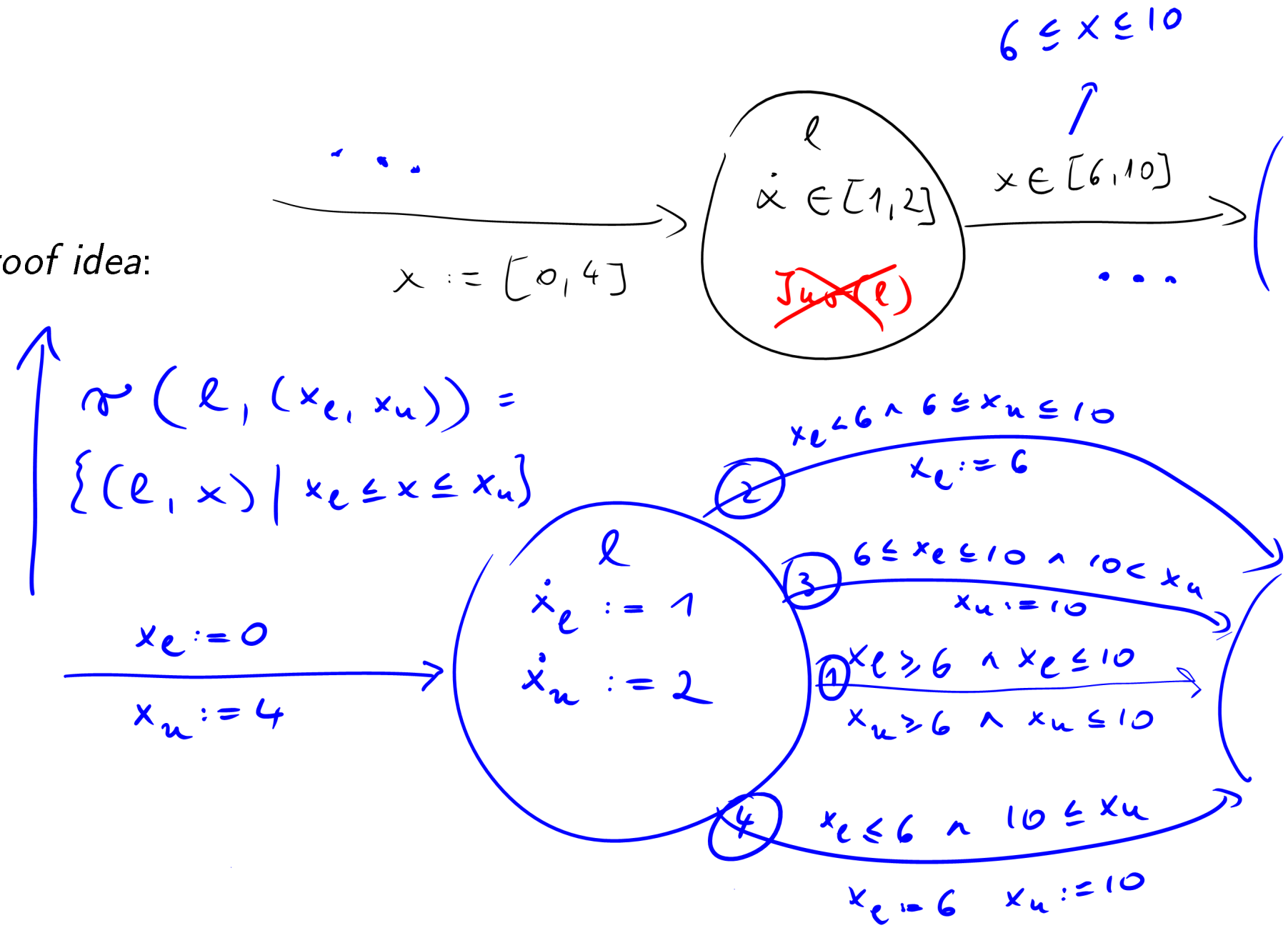
$$\begin{cases} \dot{x} = a \\ x \sim c \\ x := c \end{cases}$$

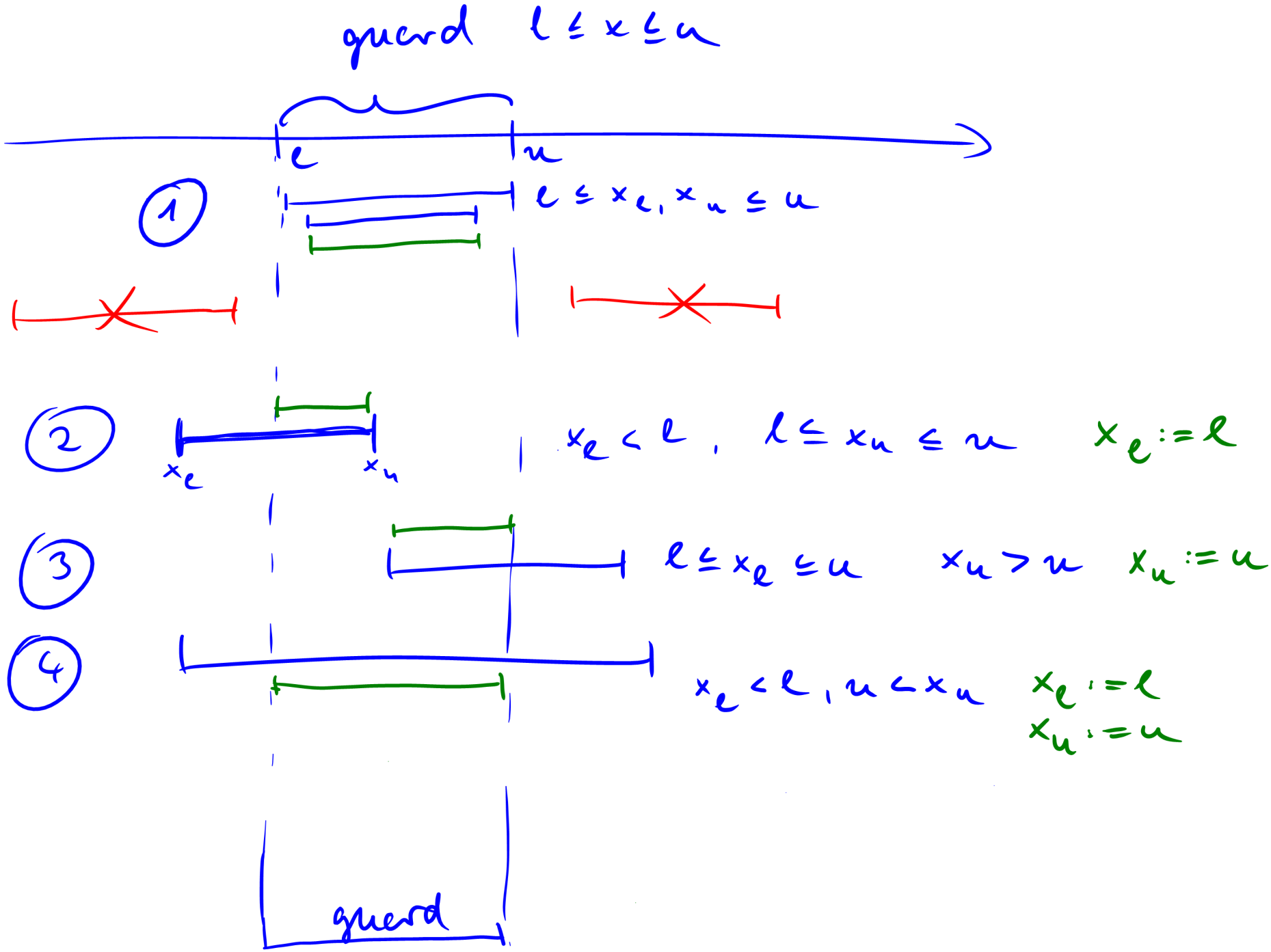
$$\begin{cases} \dot{x} \in [a,b] \\ x \in [a,b] \\ x := [a,b] \end{cases}$$

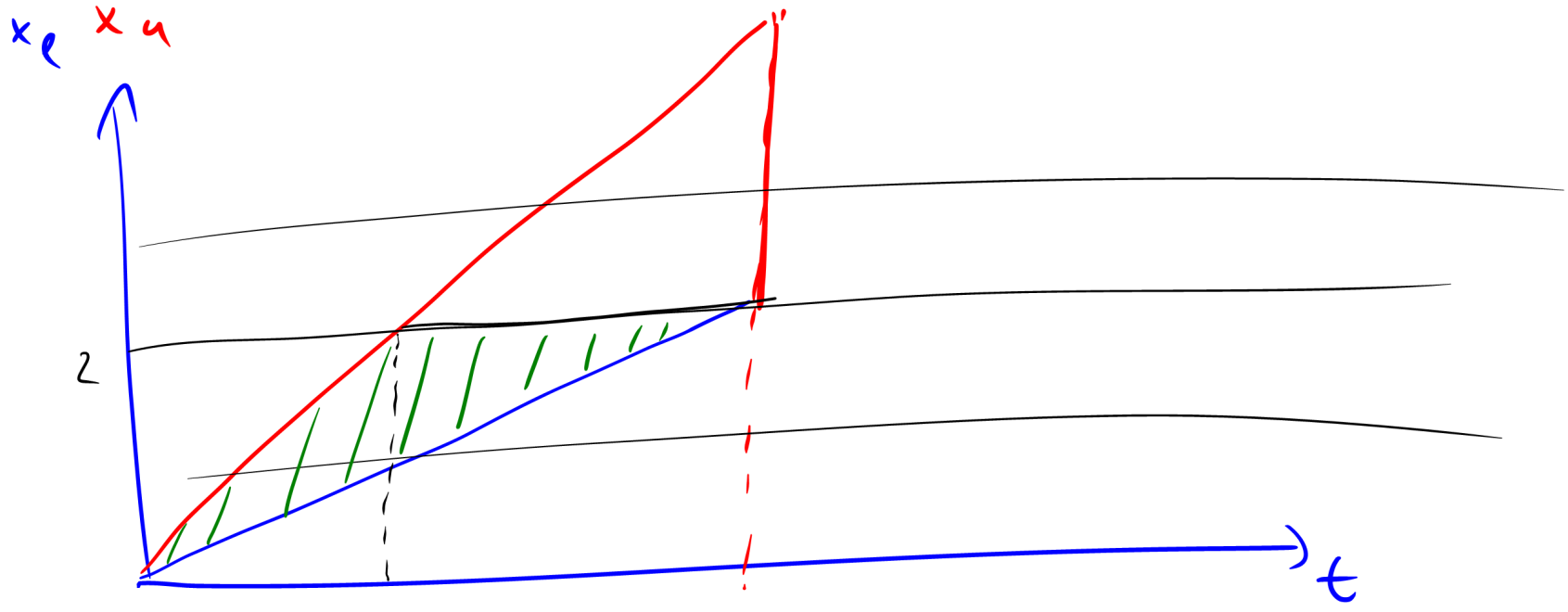
Lemma

The reachability problem for initialized rectangular automata is complete for PSPACE.

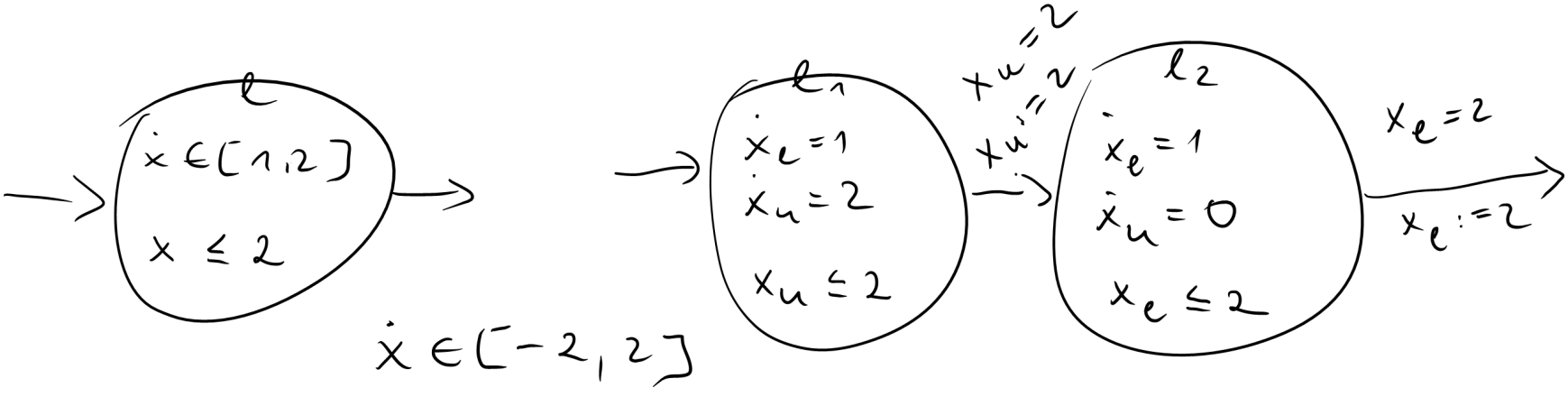
Proof idea:


6 ≤ x ≤ 10
ℓ
ẋ ∈ [1,2]
Inv(ℓ)
x ∈ [6,10]
x := [0,4]
ℜ(ℓ, (x_ℓ, x_u)) =
{(ℓ, x) | x_ℓ ≤ x ≤ x_u}
x_ℓ := 0
x_u := 4
ℓ
ẋ_ℓ := 1
ẋ_u := 2
2
x_ℓ < 6 ∧ 6 ≤ x_u ≤ 10
x_ℓ := 6
3
6 ≤ x_ℓ ≤ 10 ∧ 10 < x_u
x_u := 10
1
x_ℓ ≥ 6 ∧ x_ℓ ≤ 10
x_u ≥ 6 ∧ x_u ≤ 10
4
x_ℓ ≤ 6 ∧ 10 ≤ x_u
x_ℓ := 6 x_u := 10


guard $l \le x \le u$
l
u
1
$l \le x_l, x_u \le u$
2
x_l
x_u
$x_l < l, \; l \le x_u \le u \quad x_l := l$
3
$l \le x_l \le u \quad x_u > u \quad x_u := u$
4
$x_l < l, \; u < x_u \quad x_l := l$
$x_u := u$
guard



Invariant : $x \leq 2$



Proof idea: An n-dimensional initialized rectangular automaton A can be translated into a $2n$-dimensional initialized singular automaton B, such that B contains all reachability information about A.
The translation is similar to the subset construction for determinizing finite automata.
The idea is to replace each variable c of A by two finite-slope variables c_l and c_u: the variable c_l tracks the least possible value of c, and c_u tracks the greatest possible value of c.