Modeling and Analysis of Hybrid Systems

4. Rectangular automata

Prof. Dr. Erika Ábrahám

Informatik 2 - LuFG Theory of Hybrid Systems
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Literature

Henzinger et al.: What's decidable about hybrid automata?

Journal of Computer and System Sciences, 57:94–124, 1998

Motivation

- The special class of timed automata with TCTL is decidable, thus model checking is possible.
- What about more expressive model classes for hybrid systems?

What is decidable about hybrid automata?

Two central problems for the analysis of hybrid automata:

- Safety: The problem to decide whether something "bad" can happend during the execution of a system.
- Liveness: The problem to decide whether there is always the possibility that something "good" will eventually happen during the execution of a system.

Both problems are decidable in certain special cases, and undecidable in certain general cases.

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- all conditions, effects, and flows are described by rectagular sets.

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Definition

- A set $\mathcal{R} \subset \mathbb{R}^n$ is rectangular if it is a cartesian product of (possibly unbounded) intervals, all of whose finite endpoints are rationals.
- The set of rectangular sets in $\mathbb{R}^n$ is denoted $\mathcal{R}^n$.

Rectangular automaton

Definition

A rectangular automaton A is a tuple $\mathcal{H} = (Loc, Var, Lab, Edge, Act, Inv, Init)$ with

- finite set of locations Loc,
- finite set of real-valued variables $Var = \{x_1, \ldots, x_n\}$,
- finite set of synchronization labels Lab,
- finite set of edges $Edge \subseteq Loc \times Lab \times \mathcal{R}^n \times \mathcal{R}^n \times 2^{\{1,\ldots,n\}} \times Loc$,
- a flow function $Act : Loc \to \mathcal{R}^n$,
- an invariant function $Inv : Loc \to \mathcal{R}^n$,
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- State space: $\Sigma \subseteq Loc \times \mathbb{R}^n$ is the set of all states
- Is the state space rectangular?

Rectangular automaton

- Flows: first time derivatives of the flow trajectories in location $l \in Loc$ are within $Act(l)$
- Jumps: $e = (l, a, pre, post, jump, l') \in Edge$ may move control from location l to location l' starting from a valuation in pre, changing the value of each variable x_i to a nondeterministically chosen value from $post_i$ (the projection of $post$ to the ith dimension), such that the values of the variables $x_i \notin jump$ are unchanged.

Operational semantics

Operational semantics

$$\frac{\begin{array}{c}(l, a, \mathit{pre}, \mathit{post}, \mathit{jump}, l') \in Edge \\ \vec{x} \in \mathit{pre} \quad \vec{x}' \in \mathit{post} \quad \forall i \notin \mathit{jump}.\ x_i' = x_i \quad \vec{x}' \in Inv(l')\end{array}}{(l, \vec{x}) \xrightarrow{a} (l', \vec{x}')} \quad \texttt{Rule}_{\texttt{Discrete}}$$

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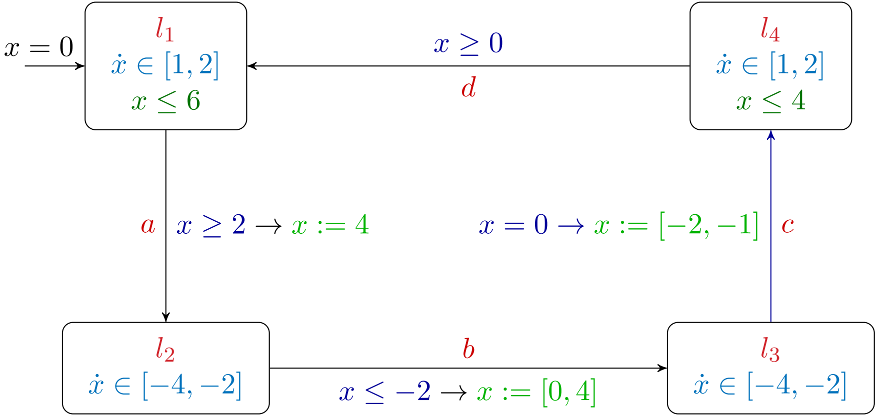
$$\frac{(t = 0 \land \vec{x} = \vec{x}') \lor (t > 0 \land (\vec{x}' - \vec{x})/t \in \mathit{Act}(l)) \quad \vec{x}' \in \mathit{Inv}(l)}{(l, \vec{x}) \xrightarrow{t} (l, \vec{x}')} \ \texttt{Rule}_{\texttt{Time}}$$

- Execution step: $\rightarrow = \xrightarrow{a} \cup \xrightarrow{t}$
- Path: $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \vec{x}_0)$, $\vec{x}_0 \in \mathit{Inv}(l_0)$
- Initial path: path $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \vec{x}_0)$, $\vec{x}_0 \in \mathit{Init}(l_0)$
- Reachability of a state: exists an initial path leading to the state

Example rectangular automaton



Remarks

- If we replace rectangular sets with linear sets, we obtain linear hybrid automata, a super-class of rectangular automata.
- A timed automaton is a special rectangular automaton.

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- A timed automaton is a special rectangular automaton.

This class lies at the boundary of decidability.

Decidability

The reachability problem is decidable for initialized rectangular automata:

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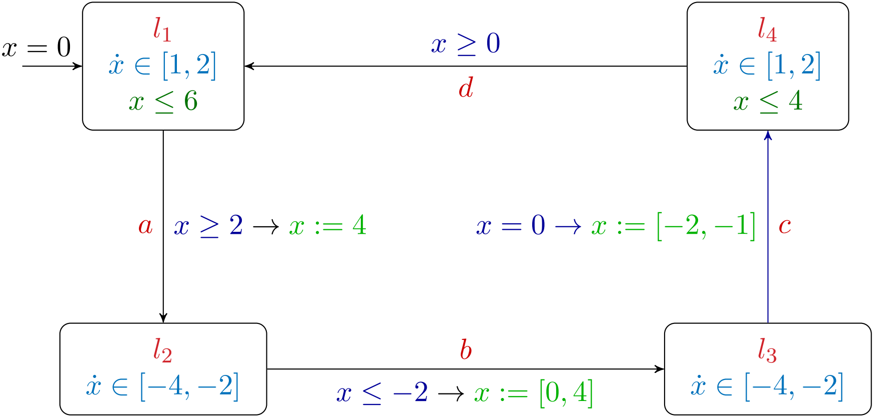
Definition

A rectangular automaton A is initialized, if for every edge $(l, a, pre, post, jump, l')$ of A, and every variable index $i \in \{1, \ldots, n\}$ with $Act(l)_i \neq Act(l')_i$, we have that $i \in jump$.

The reachability problem becomes undecidable if one of the restrictions is relaxed.

Initialized rectangular automaton



This rectangular automaton is initialized.

What we already know

A timed automaton is a special rectangular automaton such that

- $Init(l)$ is empty or a singleton for each $l \in Loc$,
- for each edge, *post*$_i$ is a single value for each $i \in$ *jump* and
- every variable is a clock, i.e., $Act(l)(x) = [1, 1]$ for all locations l and variables x.

Note: here we allow initialization and reset of clocks to any values.

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What we know:

Lemma

The reachability problem for timed automata is complete for PSPACE.

Decidability results

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The reachability problem for initialized rectangular automata is complete for PSPACE.

Decidability results

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The reachability problem for initialized rectangular automata is complete for PSPACE.

Timed automaton

$\uparrow$

Initialized stopwatch automaton

$\uparrow$

Initialized singular automaton

$\uparrow$

Initialized rectangular automaton

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

Initialized stopwatch automata

- A stopwatch is a variable with derivatives 0 or 1 only.
- A stopwatch automaton is as a timed automaton but allowing stopwatch variables instead of clocks.
- Initialized stopwatch automata can be polynomially encoded by timed automata.

Lemma

The reachability problem for initialized stopwatch automata is complete for PSPACE.

However, the reachability problem for non-initialized stopwatch automata is undecidable.

Proof idea:

Proof idea:
Assume that C is an n-dimensional initialized stopwatch automaton. Let κ be the set of constants used in the definition of C, and let $\kappa_- = \kappa \cup \{-\}$. For each $(k_1, \ldots, k_n) \in \kappa_-^n$, let $\alpha_{k_1,\ldots,k_n} : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\alpha_{k_1,\ldots,k_n}(\vec{x}) = \vec{y}$ such that $y_i = x_i$ if $k_i = -$, and $y_i = k_i$ if $k_i \neq -$.
We define an n-dimensional timed automaton D with locations $Loc_D = Loc_C \times \kappa_-^n$. Each state $\sigma = ((l, k_1, \ldots, k_n), \vec{x})$ of D represents the state $\alpha(\sigma) = (l, \alpha_{k_1,\ldots,k_n}(\vec{x}))$ of C. We extend α to state sets the natural way.
Intuitively, if the ith stopwatch of C is running (slope 1), then its value is tracked by the value of the ith clock of D; if the ith stopwatch is halted (slope 0) at value $k \in \kappa$, then this value is remembered by the current location of D.

Proof idea(continued):
The other components of D are derived from C as follows: $Var_D = Var_C$, $Lab_D = Lab_C$, $Act_D(l) = \Pi_{i=1}^{n}[1,1]$, and
$Inv_D(l, k_1, \ldots, k_n) = \alpha_{k_1,\ldots,k_n}(Inv_C(l))$.
For the initial states, assume a location l with stopwatch derivatives $d_1, \ldots, d_n$ (note: each d_i is either 1 or 0). If $Init_C(l)$ is empty then $Init_D(l, \cdot)$ are all empty. Otherwise, $Init(l)$ contains a single value $x_1, \ldots, x_n$. Let k'_i be $-$ if $d_i = 1$ and x_i otherwise. Then $Init_D(l, k'_1, \ldots, k'_n) = Init(l)$ and $Init_D(l, \cdot)$ is empty for all other cases.
Edges: exercise.
We have:

- Each state σ of C is time-abstract bisimilar to the state $\alpha(\sigma)$ of D.
- The reachable set of $Reach(C)$ of C is $\alpha(Reach(D))$.

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

$\uparrow$

Initialized singular automaton

Initialized singular automata

- A variable x_i is a finite-slope variable if $\mathit{flow}(l)_i$ is a singleton in all locations l.
- A singular automaton is as a stopwatch automaton but allowing finite-slope variables instead of stopwatches.
- Initialized singular automata can be polynomially encoded by initialized stopwatch automata.

Lemma

The reachability problem for initialized singular automata is complete for PSPACE.

Proof idea:

Proof idea: Let B be an n-dimensional initialized singular automaton and let $k_{l,i}$ denote the derivative of the ith variable in location $l \in Loc_B$ of B (i.e., $Act_B(l) = \Pi_{i=1}^{n}[k_{l,i}, k_{l,i}]$).
Let furthermore $\beta_{l,i} = 1/k_{l,i}$ if $k_{l,i} \neq 0$ and $\beta_{l,i} = 1$ otherwise.
For each location $l \in Loc_B$ of B we define a function $\beta_l : \mathbb{R}^n \to \mathbb{R}^n$ by setting $\beta_l(x_1, \ldots, x_n) = (\beta_{l,1} \cdot x_1, \ldots, \beta_{l,n} \cdot x_n)$. β_l can be viewed as a rescaling of the state space, and can be extended naturally to regions ($\beta_l(\Pi_{i=1}^{n}[v_i, v_i']) = \Pi_{i=1}^{n}[\beta_l(v_i), \beta_l(v_i')]$) and states ($\beta((l, x)) = (l, \beta_l(x))$).
We define an n-dimensional initialized stopwatch automaton C, such that all regions in the automaton B occur accordingly rescaled in C. The components of C are the same as B except the invariants $Inv_C(l) = \beta_l(Inv_B(l))$, the activities $Act_C(l) = \beta_l(Act_B(l))$, the initial regions $Init_C(l) = \beta_l(Init_B(l))$, and for each edge $e = (l, a, pre, post, jump, l') \in Edge_B$ in B there is a corresponding edge $e' = (l, a, \beta_l(pre), \beta_{l'}(post), jump, l') \in Edge_C$ in C.
We have:

- Each state σ of B is time-abstract bisimilar to the state $\beta(\sigma)$ of C.
- The reachable set of $Reach(B)$ of B is $\beta(Reach(C))$.

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

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Initialized singular automaton

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Initialized rectangular automaton

Lemma

The reachability problem for initialized rectangular automata is complete for PSPACE.

Proof idea:

Proof idea: An n-dimensional initialized rectangular automaton A can be translated into a $2n$-dimensional initialized singular automaton B, such that B contains all reachability information about A.
The translation is similar to the subset construction for determinizing finite automata.
The idea is to replace each variable c of A by two finite-slope variables c_l and c_u: the variable c_l tracks the least possible value of c, and c_u tracks the greatest possible value of c.