Satisfiability Checking

Interval Constraint Propagation

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Informatik 2
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Motivation

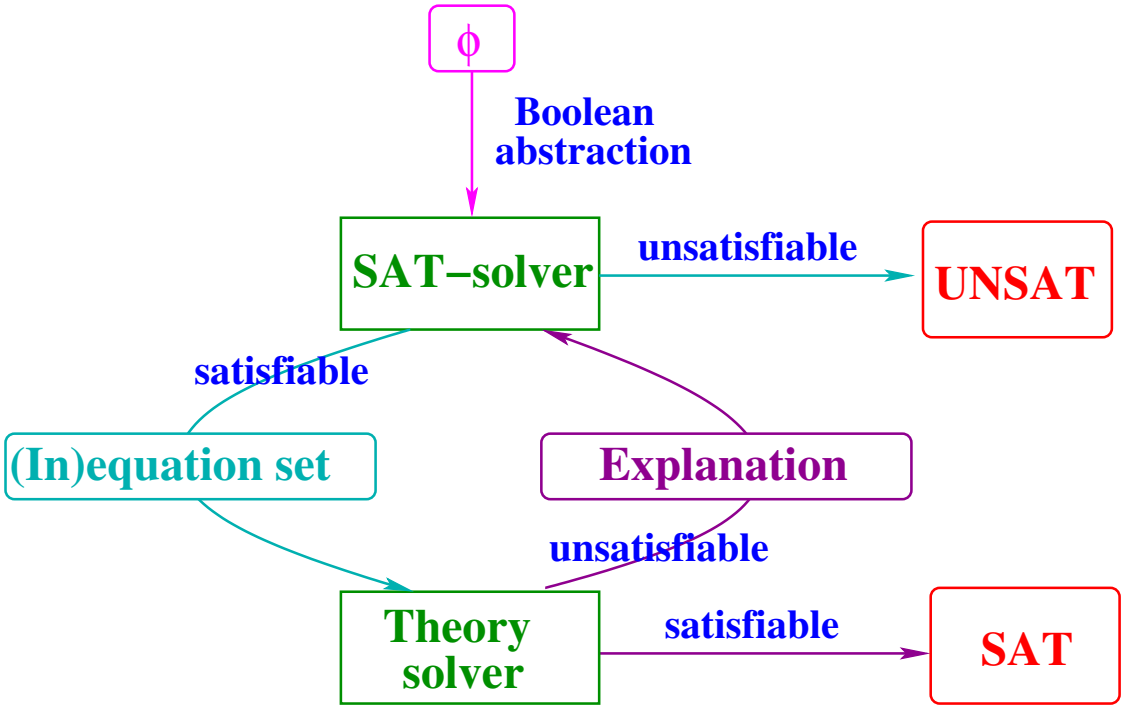
We consider input formulae φ from the theory of *quantifier-free nonlinear real arithmetic* (QFNRA):

$$p := r \mid x \mid (p + p) \mid (p - p) \mid (p \cdot p)$$
$$c := p < 0 \mid p = 0$$
$$\varphi := c \mid (\varphi \wedge \varphi) \mid \neg\varphi$$

$r \in \mathbb{Q}, x \in \mathit{Var}(\varphi)$

- Solving QFNRA has exponential complexity $\rightarrow$ hard to solve
- Current approaches for solving QFNRA:
  - Virtual substitution (VS)
  - Cylindrical algebraic decomposition (CAD)
  - Gröbner bases
  - ICP
- Procedures for linear real arithmetic (LRA) solving: Simplex

ICP



Interval constraint propagation [HMK97]

- Incomplete method
- Cheap reduction of search space
- Requires initial box

Interval

Definition (Interval)

Interval $I = [l_I; u_I] = \{x \in \mathbb{R} | l_I \sim x \sim u_I\}, l_I, u_I \in \mathbb{R}$, where l_I is called the lower bound and u_I the upper bound of I.

Possible bound types:

- Weak ($l_I \leq x \leq u_I$)
- Strict ($l_I < x < u_I$)
- Infinity ($-\infty < x < +\infty$)
- Combinations

Definition (Interval box)

N-dimensional box $B \in \mathbb{I}^n$.

Basic Interval arithmetic

Goal: Extend real arithmetic to intervals [Kul08].

- Coefficients can be intervals
- Operations maintain all correct solutions

Definition (Interval addition)

$A, B \in \mathbb{I} : A + B = [l_a + l_b; u_a + u_b]$

Example (Addition)

$[-1; 5] + [1; 4] = [0; 9]$

$[-2; 3] + 4 = [-2; 3] + \underbrace{[4; 4]}_{[4]} = [2; 7]$

Basic Interval arithmetic

Definition (Interval subtraction)

$A, B \in \mathbb{I} : A - B = [l_a - u_b; u_a - l_b]$

Example (Subtraction)

$[-1; 5] - [1; 4] = [-5; 4]$
$[-2; 3] - 4 = [-2; 3] + [4] = [-6; -1]$

Definition (Interval multiplication)

$A, B \in \mathbb{I} : A \cdot B =$
$[min(l_a \cdot l_b, l_a \cdot u_b, u_a \cdot l_b, u_a \cdot u_b); max(l_a \cdot l_b, l_a \cdot u_b, u_a \cdot l_b, u_a \cdot u_b)]$

Example (Multiplication)

$[-1; 5] \cdot [1; 4] = [-4; 20]$
$[-2; 3] \cdot 4 = [-2; 3] + [4] = [-8; 12]$

Basic Interval arithmetic

Special case of multiplication: Square

Definition (Interval square)

Squaring an interval can only result in positive values:
$A \in \mathbb{I} : A^2 = (A \cdot A) \cap [0; +\infty)$

Example (Square)

$[-1; 5]^2 = [0; 25]$

Definition (Interval division ($0 \notin B$))

$A, B \in \mathbb{I} : A \div B = A \cdot \frac{1}{B} = A \cdot [\frac{1}{u_b}; \frac{1}{l_b}]$

Example (Division)

$[2; 3] \div [4; 5] = [2; 3] \cdot \frac{1}{B} = [2; 3] \cdot [\frac{1}{5}; \frac{1}{4}] = [\frac{2}{5}; \frac{3}{4}]$

Extended Interval arithmetic

Problem: Division (divisor interval may contain 0):

Example (Division by zero)

$[1;3] \div [-2;3] = ?$
Usual approach:
$[1;3] \div [-2;3] = [1;3] \cdot [\frac{1}{3}; -\frac{1}{2}] \rightarrow$ invalid bounds (let's ignore this)
$[1;3] \cdot [\frac{1}{-2}; \frac{1}{3}] = [-\frac{3}{2}; 1]$ but: $\underbrace{\frac{1}{2}}_{\in[1;3]} \div \underbrace{\frac{1}{8}}_{\in[-2;3]} = \underbrace{4}_{\notin[-\frac{3}{2};1]}$

Introduce new rules for extended interval division [Kul08] such that:
$[1;3]/[-2;3] = (-\infty; \frac{1}{-2}] \cup [\frac{1}{3}; +\infty)$

Propagation

Propagation of intervals can imply new bounds

- Insert current bounds for all except one variable in one constraint
- Apply interval arithmetic to gain new bounds
- Update bounds via intersection

Example (Propagation)

$x \in [1;3], y \in [1,2], c_1 : y = x, c_2 : y = x^2$
$c_2, x : x = \sqrt{y} \rightarrow x = \sqrt{[1;2]} = [1;\sqrt{2}] \rightarrow x \in [1;3] \cap [1;\sqrt{2}] = [1;\sqrt{2}]$
$c_1, y : y = x \rightarrow y = [1;\sqrt{2}] \rightarrow y \in [1;2] \cap [1;\sqrt{2}] = [1;\sqrt{2}]$

Contraction sequence:
$x : [1;2] \overset{c_2,x}{\rightarrow} [1;\sqrt{2}] \overset{c_2,x}{\rightarrow} [1;\sqrt[4]{2}] \overset{c_2,x}{\rightarrow} [1;\sqrt[8]{2}] \overset{c_2,x}{\rightarrow} \ldots \overset{c_2,x}{\rightarrow} [1;1]$
$y : [1;3] \overset{c_1,y}{\rightarrow} [1;\sqrt{2}] \overset{c_1,y}{\rightarrow} [1;\sqrt[4]{2}] \overset{c_1,y}{\rightarrow} [1;\sqrt[8]{2}] \overset{c_1,y}{\rightarrow} \ldots \overset{c_1,y}{\rightarrow} [1;1]$

We define a pair of constraint and variable as a contraction candidate (CC).

Preprocessing

Propagation requires constraints to be solvable in one variable
$\rightarrow$ Preprocessing required: Linearization.

- Separate linear and nonlinear constraints
- Introduce new variables for nonlinear monomials
- Obtain two sets: Linear constraints (including linearizations) and nonlinear monomials

Preprocessing

Toy example:

$$x^2 \cdot y + z = 0$$

$$v_1 + z = 0 \wedge v_1 = x^2 \cdot y$$

Algorithm

Setup:

- Input: Conjunction of constraints
- Variables bounded by intervals (initial box)

Goal

Reduce the size of the input box to a specified diameter while guaranteeing that a solution (if available) is still contained inside the box.

Note: In this context we pass the gained box (solution candidate) to a backend implementing a complete method.

- We stop contraction when some diameter is reached
- Complete methods profit from reduced search space

Algorithm

```
Answer icp(Box init, ConstraintSet constraints):
  Box current = init;
  bool finished = false;
  while(!finished){
    bool done = false;
    while(!done) {
      CC candidate = chooseCandidate(constraints);
      done = contract(current, candidate); // CBR: current
    }
    if(!empty(current))
      if(callBackend(current) == SAT)
        return SAT;
    else
      finished = chooseBox(current);
  }
  return UNSAT;
```

Algorithm

Setup:

- Conjunction of constraints
- Variables bounded by intervals (initial box)
- Goal: Reduce box for complete methods (backend)

Requirements:

- Method to choose CCs
- Method for contraction
- Mechanism to keep track of boxes
- Mechanism to choose next box

Contraction

General approach: Contract via propagation.
Problems:

- Contraction is in general not predictable
- Contraction may stop before target diameter reached
- Contraction may cause a split (heteronomous split)

Example (Contraction candidate choice)

Consider $\{c_1 : y = x, c_2 : y = x^2\}$ with initial intervals $I_x := [1, 3]$ and $I_y := [1, 2]$
At each step we can consider 4 contractions:

- $I_x \overset{c_1, x}{\rightarrow} [1, 2]$ (33% reduction)
- $I_y \overset{c_1, y}{\rightarrow} [1, 2]$ (0% reduction)
- $I_x \overset{c_2, x}{\rightarrow} [1, \sqrt{2}]$ (47% reduction)
- $I_y \overset{c_2, y}{\rightarrow} [1, 2]$ (0% reduction)

$\rightarrow$ Contraction gain varies.

Reinforced learning

We can improve the choice of CCs by heuristics:

- The algorithm selects the next contraction candidate with the highest weight $W_k^{(ij)} \in [0; 1]$.
- Afterwards the weight is updated (according to the relative contraction $r_{k+1}^{(ij)} \in [0; 1]$).

Weight updating:

$$W_{k+1}^{(ij)} = W_k^{(ij)} + \alpha(r_{k+1}^{(ij)} - W_k^{(ij)})$$

The factor $\alpha \in [0; 1]$ decides how the importance of the events is rated:

- Large α (e.g. 0.9) $\rightarrow$ The last recent event is most important
- Small α (e.g. 0.1) $\rightarrow$ The initial weight is most important

CCs with a weight less than some threshold ε are not considered for contraction.

Contraction

Requirements for contraction operator:

- Does not drop solutions
- Detects if no solution is contained in the current box

The second usually cannot be guaranteed.

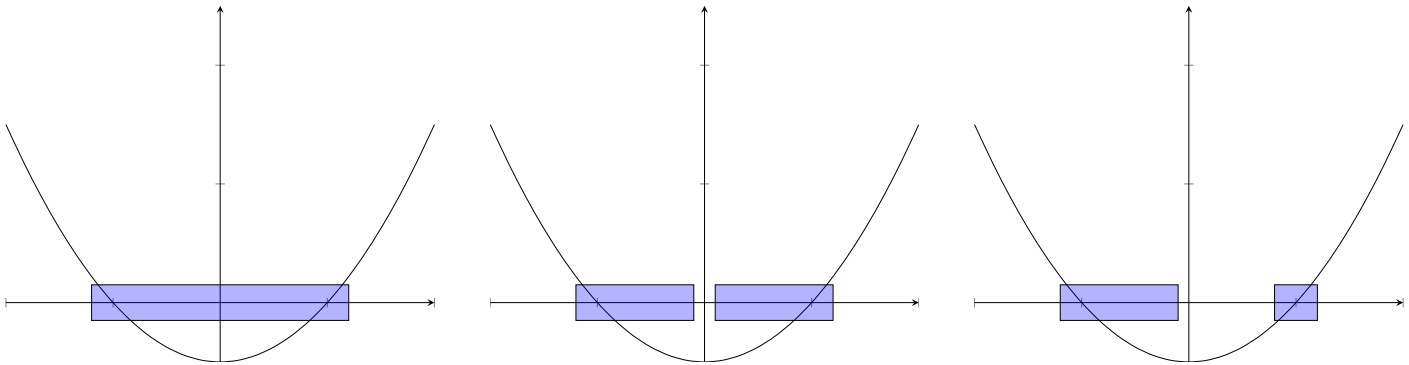
Usual contraction operators:

- Propagation (original contractor)
- Interval Newton method

General approach: Contract as long as contraction gain is large enough.

Splitting

When the weight of all CCs is below the threshold we do not make progress $\rightarrow$ split manually (autonomous split).



Problems:

- How to store boxes
- How to select the next box

History Tree

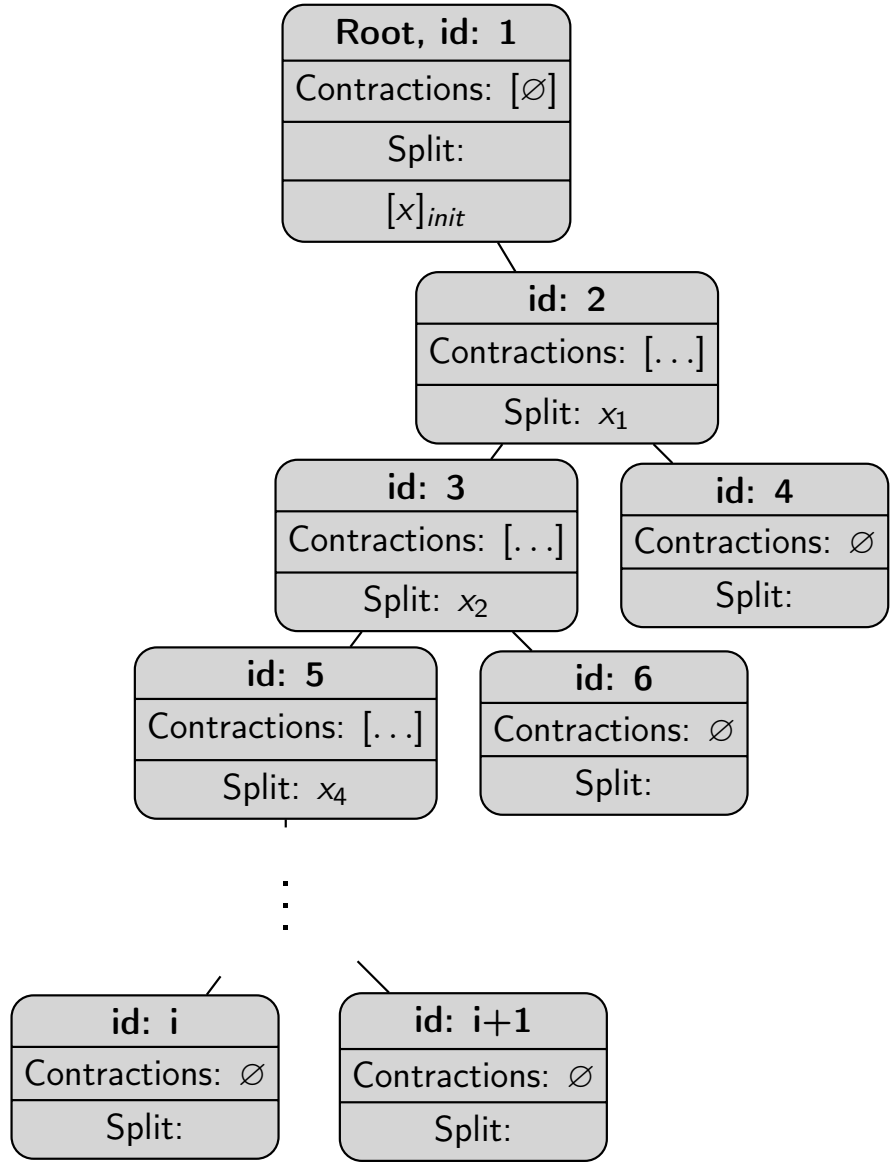
To keep track of current status we utilize a tree-structure, which holds solver states:

- Search box
- Applied contractions
- Splitting dimension

Additionally we can use the tree to collect infeasible subsets:

1. Infeasible box $\rightarrow$ propagate reasons to parent
2. (optional) skip boxes

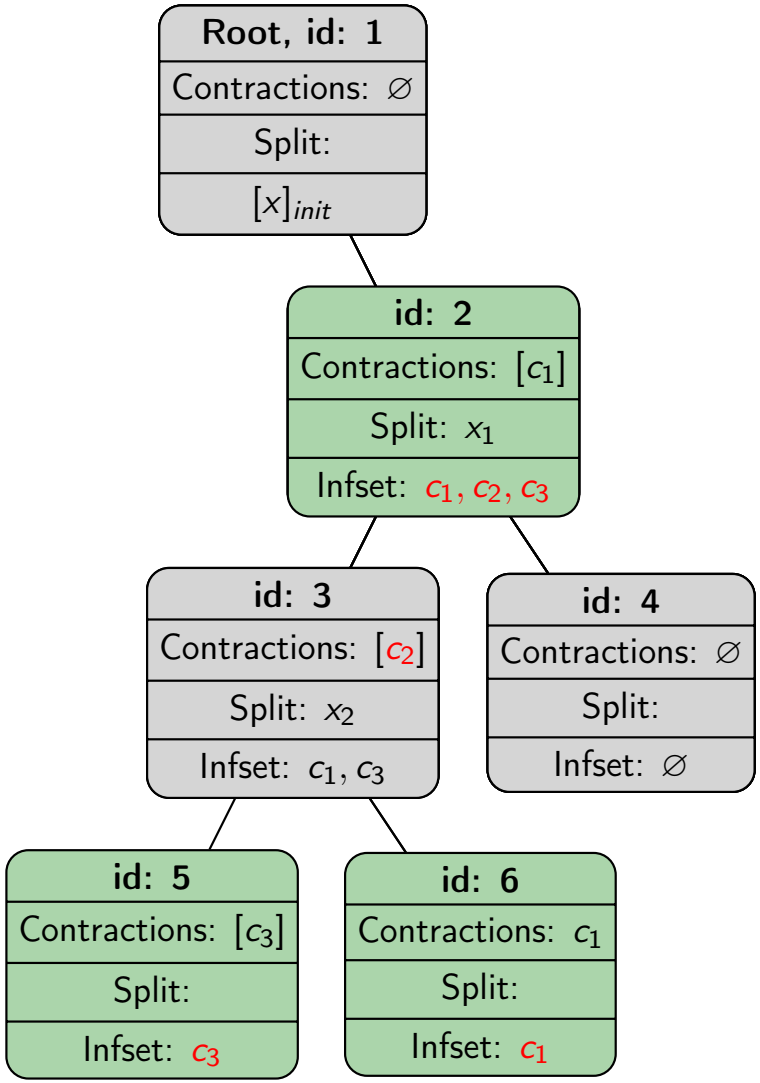
$\rightarrow$ We generate a set of constraints which includes the infeasible subset.



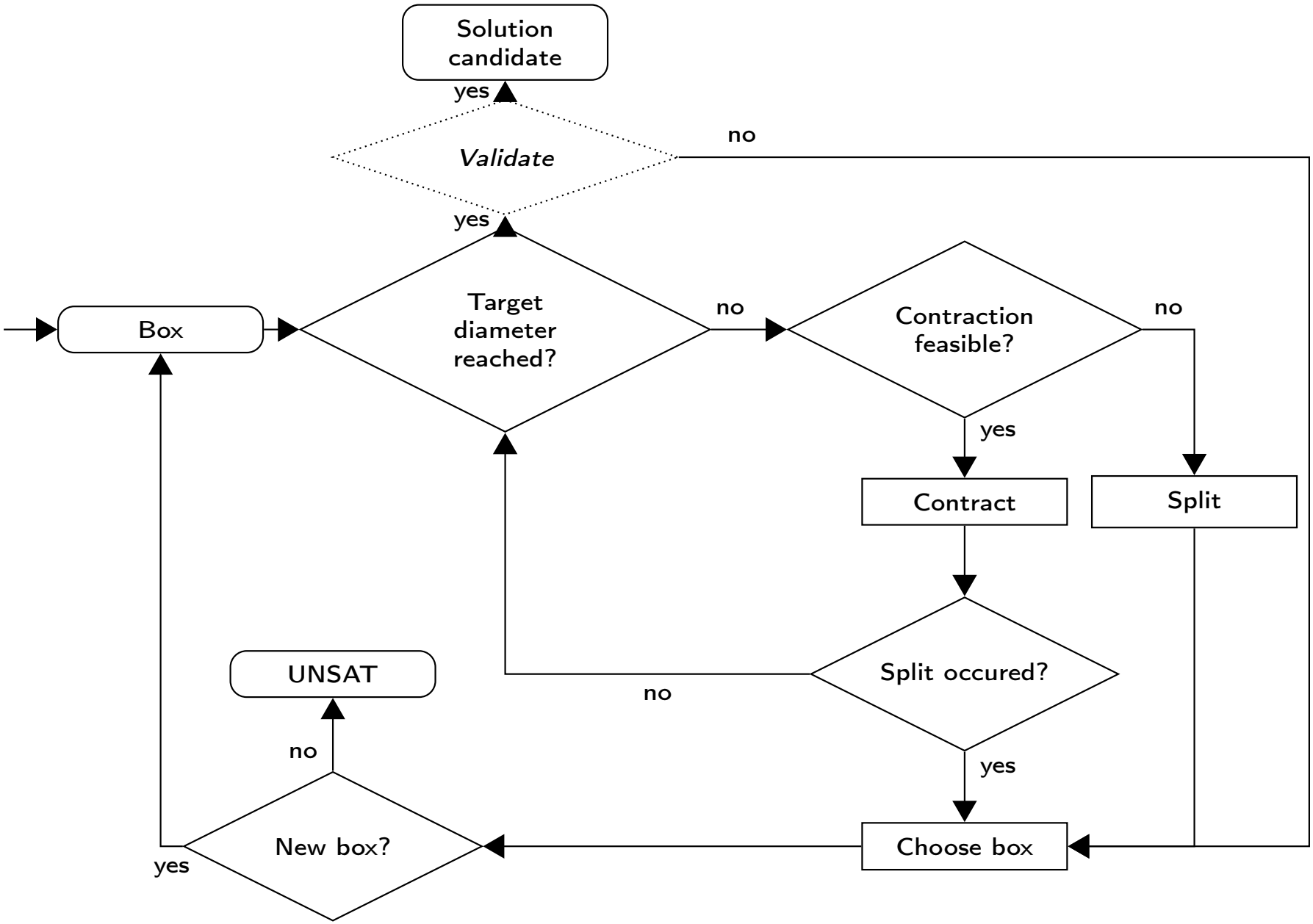
If no further heuristics is applied we traverse the tree pre-order.

History tree example

Setup: constraints $c_1, \ldots, c_4$, variables $x_1, \ldots, x_3$



Algorithm overview



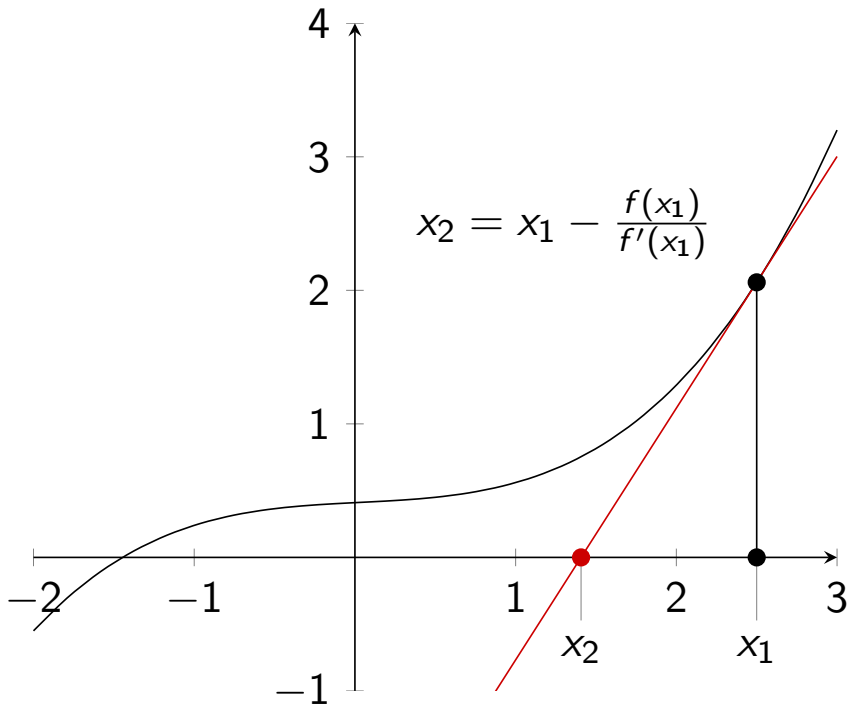
Improvements

Additional improvements:

- More sophisticated contraction (Interval Newton method [MKC09])
- Use linear solver for linear constraints
- Introduce box validation
- Involve SAT-Solver for box-choice
- Introduce splitting heuristics

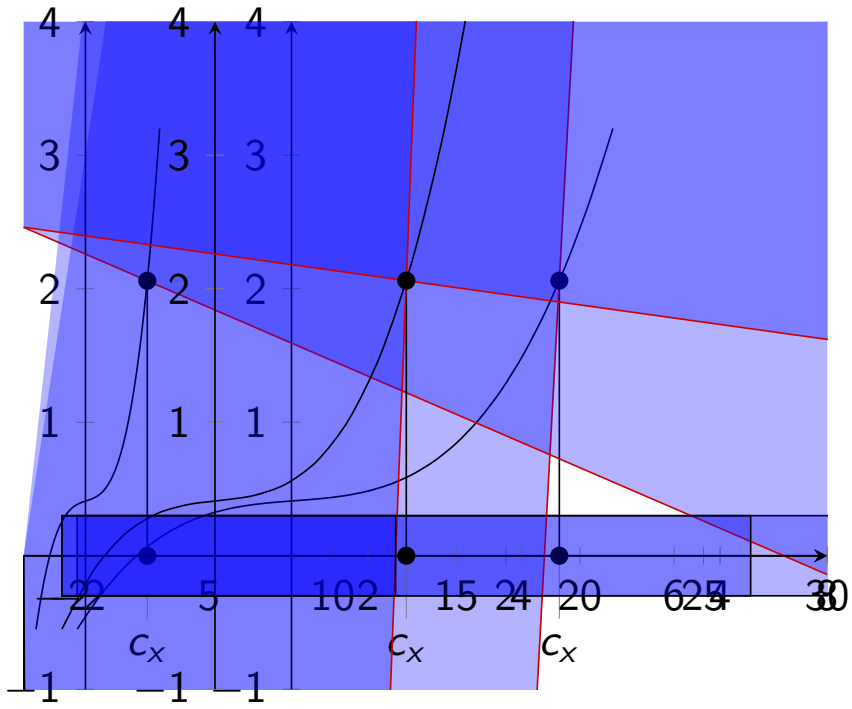
Interval Newton method

Reminder: Newton method for root finding (univariate polynomials):



Interval Newton method

Interval extension of Newtons method [HR97]:



$f(x) = 0.1 \cdot x^3 - 0.01 \cdot (x-3)^2 + 0.5$ Starting interval: $x \in [-2; 7]$, sampling point: $c_x = 2.5$ Tangent:
$[2.5] - \frac{f(c_x)}{f'(x)} = (-\infty; 2.36018] \cup [28.25; +\infty)$ New interval:
$x \in [-2; 7] \cap (-\infty; 2.36018] \cup [28.25; +\infty) = [-2; 2.36018]$

Interval Newton method

Componentwise Newton operator

$$N_{cmp}(\overbrace{[x]}^{box}, \overbrace{f_i(x_1,\ldots,x_n), j}^{CC}) :=$$
$$c([x]_{x_j}) - \frac{f_i([x]_{x_1},\ldots,[x]_{x_{j-1}}, c([x]_{x_j}), [x]_{x_{j+1}},\ldots,[x]_{x_n})}{\frac{\partial f_i}{\partial x_j}([x]_{x_1},\ldots,[x]_{x_n})}$$

Reminder(Multivariate Newton):

$N_{cmp}(x, f_i(x_1,\ldots,x_n), j) = x - \frac{f_i(x)}{\frac{\partial f_i}{\partial x_j}(x)}$ The operator N_{cmp} has two

important properties:

- If x^* is a solution and $x^* \in [x]$, then $x^* \in N_{cmp}([x], i, j)$
- If $[x] \cap N_{cmp}([x], i, j) = \varnothing$, there is no solution in $[x]$

$\rightarrow$ Advantage: No diverging behavior like the original Newton method due

Introduce linear solver

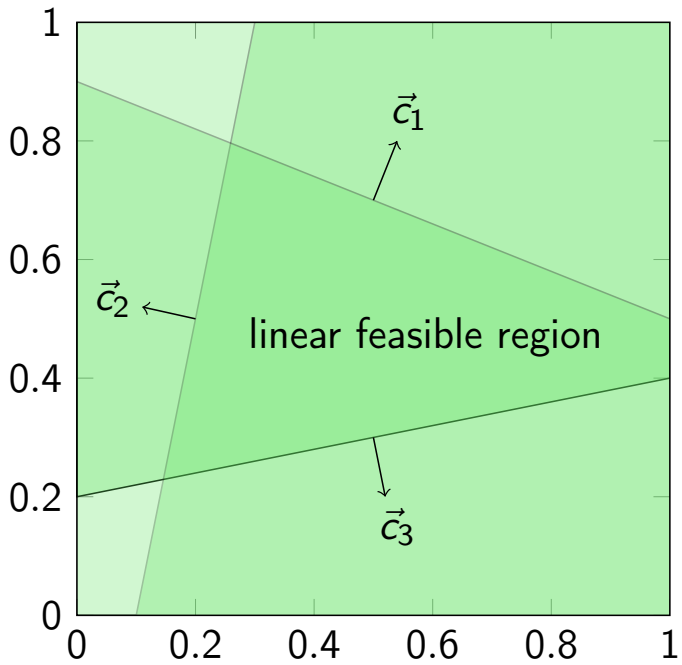
ICP is not well-suited for linear problems
($\rightarrow$ slow convergence phenomenon).

Make use of linear solvers for linear constraints:

- Pre-process to separate linear and nonlinear constraints
- Determine linear feasible region
- Use nonlinear constraints for contraction
- Validate resulting boxes against linear feasible region
- In case box is linear infeasible: Add violated linear constraint for contraction

Validation

- Resulting intervals represent n–dimensional hyperboxes
- Boxes may violate linear constraints
- Check resulting boxes against the linear feasible region

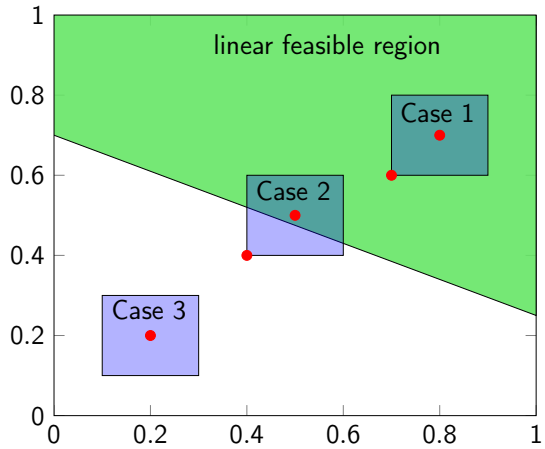


Case distinction

We have to distinct 3 cases:

- Case 1: The resulting interval lies completely inside the linear feasible region
- Case 2: The hyperbox partly covers the linear feasible region
- Case 3: The solution interval lies outside the linear feasible region

The problem lies in separating the 1st case from the 2nd one.



If a checked point is valid $\rightarrow$ return SAT

Raise splitting

Instead of having an internal box management (e.g. via a tree) we can raise a split to the SAT solver

- Create deductions (tautologies)
- Deduction is added as additional information to the SAT solver
- SAT solver decides based on its information which split to take

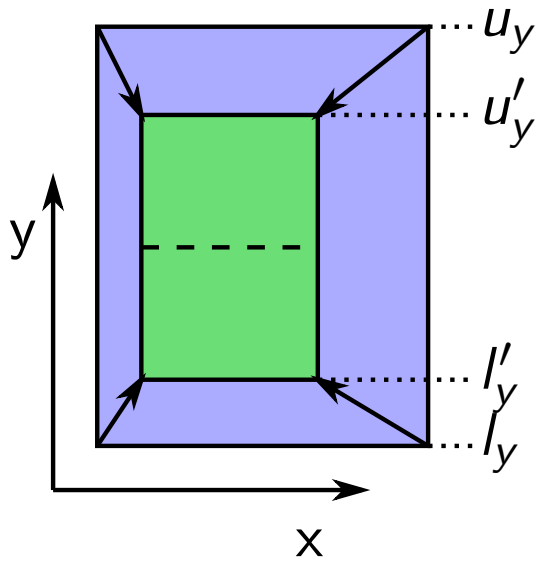
ICP deductions include:

- The split
- (optional) A premise in form of previous contractions

Raise splitting

Setup:

- Original box (blue) contracted to new box (green)
- Contraction via constraints $c_i, i = 1, \ldots, n$
- Demanded split at $c(l_y)$



We create a splitting deduction:

$$\underbrace{x \in [l_x] \wedge y \in [l_y] \wedge (c_1 \wedge \ldots \wedge c_n)}_{\textit{premise}} \rightarrow \underbrace{(y \in [l'_y; c(l_y)] \oplus y \in (c(l_y); u'_y])}_{\textit{split}}$$

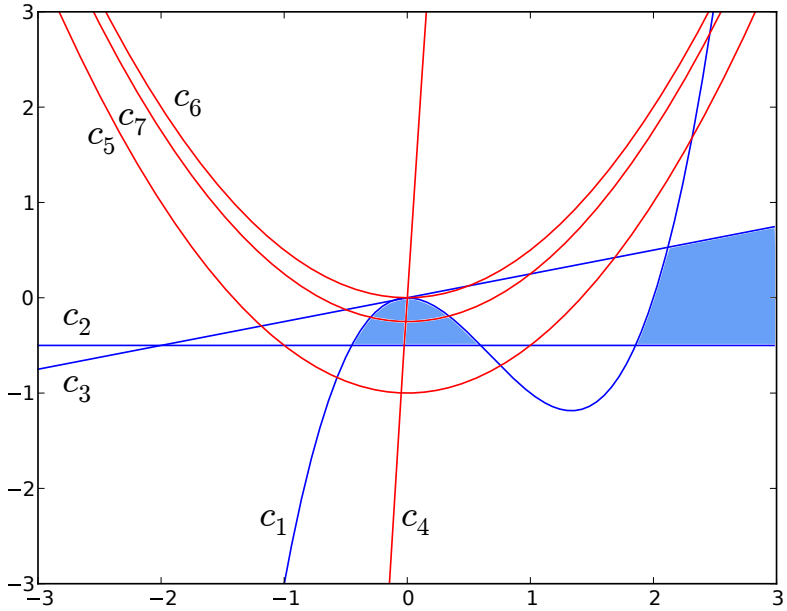
Experimental results

Results of a prototype implementation:

- Linear and nonlinear constraints
- Weak and strict inequations

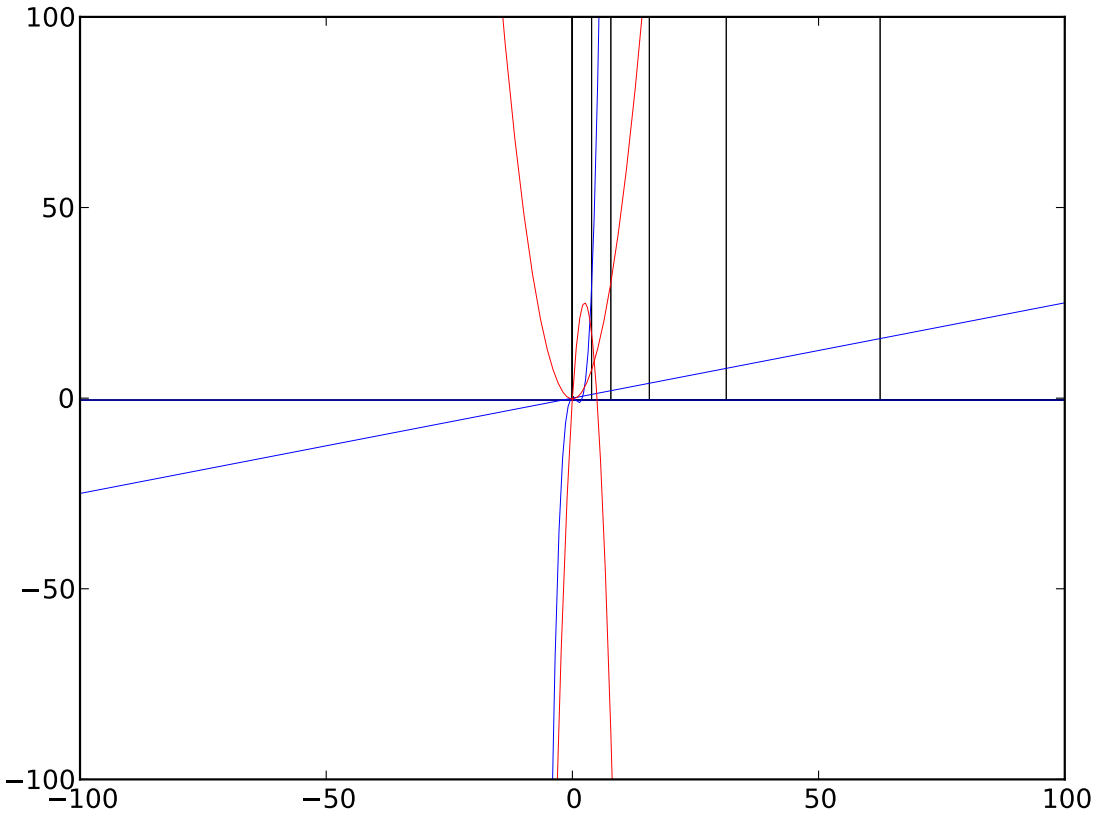
Test example:

$$\left[\left(c_1 : x^3 - 2x^2 > y \right) \right.$$
$$\wedge \left(c_2 : -\frac{1}{2} \leq y \right)$$
$$\wedge \left(c_3 : \frac{1}{4}x \geq y \right)$$
$$\wedge \left(c_4 : -4x^2 + 20x = y \vee c_5 : \frac{1}{2}x^2 - 1 = y \right)$$
$$\left. \wedge \left(c_6 : x^2 = y \vee c_7 : \frac{1}{2}x^2 - \frac{1}{4} = y \right) \right]$$



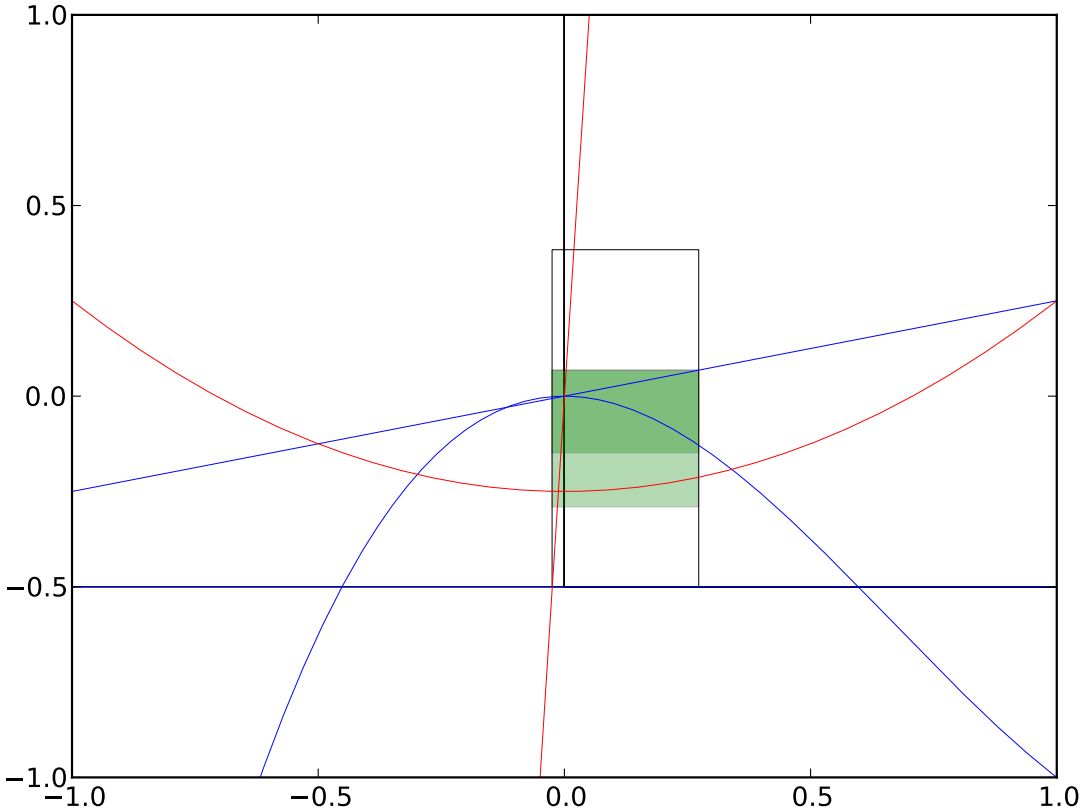
Experimental results

target size = 1



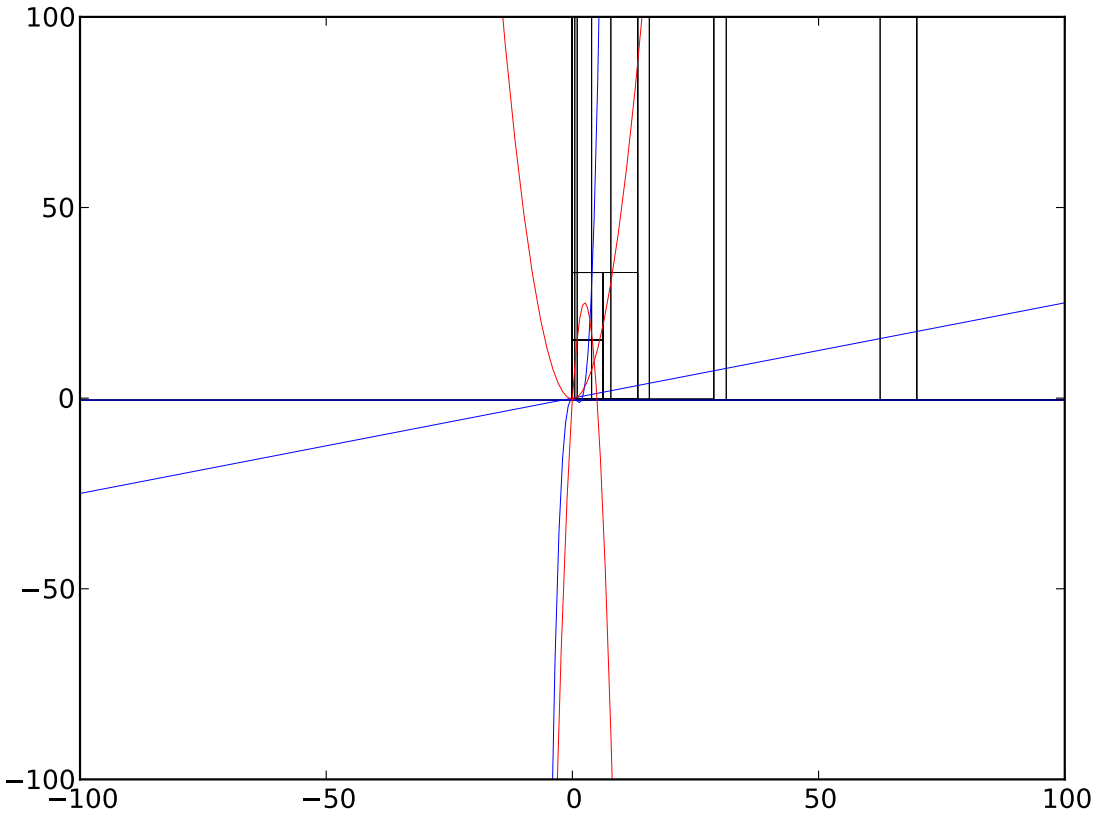
Experimental results

target size = 1



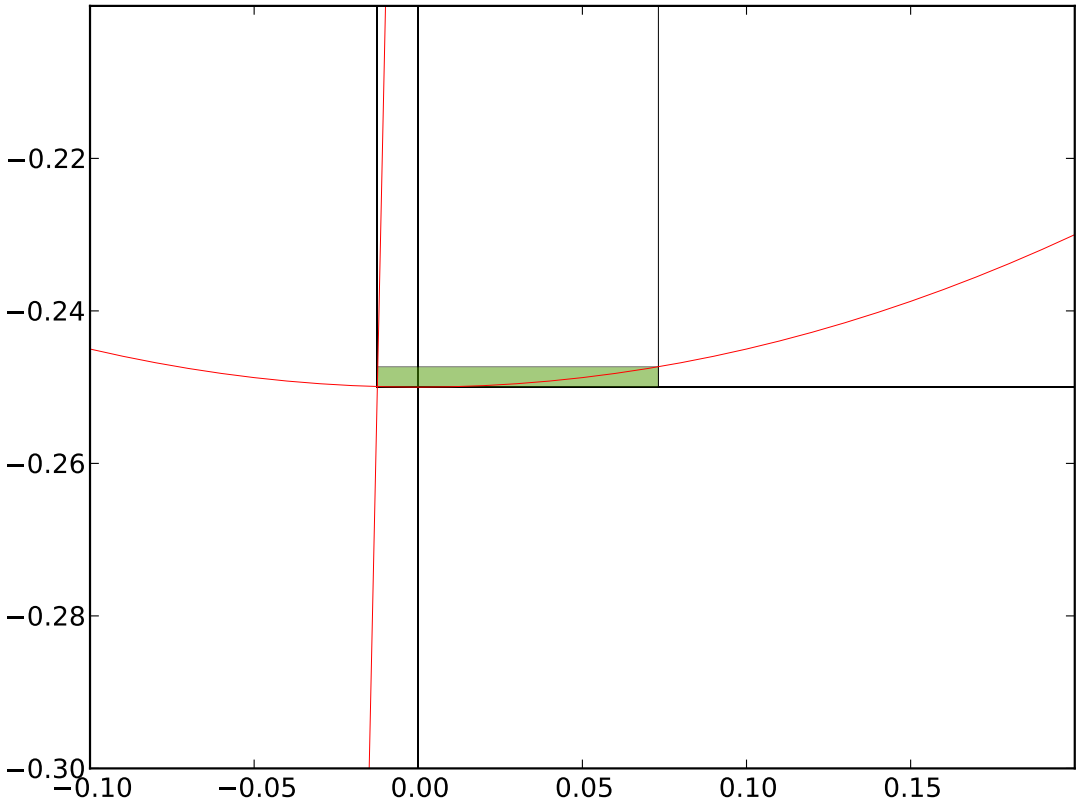
Experimental results

target size = 0.1



Experimental results

target size = 0.1



Experimental results

∅	backend calls	inv. boxes	depth avg(max)	# contr.	# splits
10	8	1	28(64)	155	100
1	7	2	25(66)	219	120
0.1	4	5	36(82)	369	177
0.01	7	9	39(94)	525	218
0.001	4	17	43(106)	638	247
0.0001	4	22	48(122)	735	281

Summary

Aspects relevant for exercise and exam:

- Basic interval arithmetic (no division by intervals containing 0)
- Interval propagation
- Problems during contraction
- Splitting
- Basic algorithm (see Slide 20)
- Possible extensions (not in detail)

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