Examples for the Omega test

1 The $\widehat{mod}$ operator

Before the Omega test can be applied, we need to eliminate all equations. This is done using the special modulo operator $\widehat{mod}$ defined as

$$a \mathbin{\widehat{mod}} b = a - b\lfloor a/b + 1/2 \rfloor \ .$$

The $\widehat{mod}$ operator differs from the usual modulo in that the function values of $(\cdot \mathbin{\widehat{mod}} b)$ are from the interval $[-b/2, b/2)]$ instead of $[0, b)$. For example, $(\cdot \mathbin{\widehat{mod}} 4)$ defines

$$\begin{array}{lll}
-4 \mathbin{\widehat{mod}} 4 & = -4 - \lfloor -4/4 + 1/2 \rfloor & = 0 \\
-3 \mathbin{\widehat{mod}} 4 & = -3 - \lfloor -3/4 + 1/2 \rfloor & = 1 \\
-2 \mathbin{\widehat{mod}} 4 & = -2 - \lfloor -2/4 + 1/2 \rfloor & = -2 \\
-1 \mathbin{\widehat{mod}} 4 & = -1 - \lfloor -1/4 + 1/2 \rfloor & = -1 \\
0 \mathbin{\widehat{mod}} 4 & = 0 - \lfloor 0/4 + 1/2 \rfloor & = 0 \\
1 \mathbin{\widehat{mod}} 4 & = 1 - \lfloor 1/4 + 1/2 \rfloor & = 1 \\
2 \mathbin{\widehat{mod}} 4 & = 2 - \lfloor 2/4 + 1/2 \rfloor & = -2 \\
3 \mathbin{\widehat{mod}} 4 & = 3 - \lfloor 3/4 + 1/2 \rfloor & = -1 \\
4 \mathbin{\widehat{mod}} 4 & = 4 - \lfloor 4/4 + 1/2 \rfloor & = 0
\end{array}$$

We use this $\widehat{mod}$ operator to make the absolute value of coefficients in equations smaller. Since the absolute values for the $\widehat{mod}$ operator are smaller than those defined by the standard modulo, $\widehat{mod}$ is better suited for this task.

2 First example for the elimination of equations

If our problem contains equations then we would like to use them for variable elimination. If there is an equation in which one of the variables, say x_i, has a coefficient 1 or -1 then we can use this equation to eliminate x_i. If there are no such equations then we apply a procedure to generate new equations with smaller coefficients until one of the coefficients get 1 or -1. Note that

we cannot apply division to the equations, since it would result in non-integer coefficients.
We determine an equation $\sum_{i=1}^{n} a_i x_i = b$ (a_i and b are constants, x_i are variables) in which a coefficient $a_i \neq 0$ with the smallest absolute value appears. We apply $(\cdot \widehat{mod}\ (a_i+1))$ to each variable coefficient on the left-hand side of the equation. Thereby we modify the value of the left-hand side by a multiple of (a_i+1). We apply $(\cdot \widehat{mod}\ (a_i+1))$ also to the constant b on the right-hand side and add the term $(a_i+1)\sigma$ to it to get an equation with reduced coefficients, where σ is a fresh variable. We add this new equation to our problem and reply the procedure until there is a coefficient with absolute value 1 in the new equation. In case the new equation satisfies this condition we use it for variable elimination.
Assume the equation

$$4x + 3y = 7\ .$$

In this example the smallest coefficient is 3, therefore we apply $(\cdot \widehat{mod}\ 4)$ to generate the new equation

$$\begin{aligned}(4 \widehat{mod}\ 4)x + (3 \widehat{mod}\ 4)y &= 4\sigma + (7 \widehat{mod}\ 4)\\ 0 \cdot x + -1 \cdot y &= 4\sigma + (-1)\\ y &= -4\sigma + 1\end{aligned}$$

We add $y = -4\sigma + 1$ to our problem and use it to eliminate y in $4x + 3y = 7$:

$$\begin{aligned}4x + 3(-4\sigma + 1) &= 7\\ 4x - 12\sigma + 3 &= 7\\ 4x - 12\sigma &= 4\\ x - 3\sigma &= 1\\ x &= 3\sigma + 1\end{aligned}$$

Our problem has now the single equation $x = 3\sigma + 1$ which we use to eliminate x in the empty set of remaining constraints. The result of the elimination gives back the satisfaction of the problem.
We have a satisfying solution for each value of σ:

σ	...	-3	-2	-1	0	1	2	3	...
x	...	-8	-5	-2	1	4	7	10	...
y	...	13	9	5	1	-3	-7	-11	...

3 Second example for the elimination of equations

Assume now that we extend the problem from the previous section with the requirement that x and y are non-negative:

$$\begin{aligned} 4x + 3y &= 7 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

We start to solve the problem as above: we use the equation to generate a new one $y = -4\sigma + 1$. We use this equation to eliminate y. For $y > 0$ this gives us $4\sigma \leq 1$ from which we get by tightening $\sigma \leq 0$. The problem is now specified by

$$\begin{aligned} x &= 3\sigma + 1 \\ x &\geq 0 \\ \sigma &\leq 0 \end{aligned}$$

The equation $x = 3\sigma + 1$ is suited to eliminate x:

$$\begin{aligned} 3\sigma &\geq -1 \\ \sigma &\leq 0 \end{aligned}$$

We get by tightening

$$\begin{aligned} \sigma &\geq 0 \\ \sigma &\leq 0 \end{aligned}$$

We can directly see that the only solution is $\sigma = 0$, $x = 1$ and $y = 1$.

4 Example for the Omega test

Assume that we want to organize a game evening. We can arrange three types of games. The number of players per game are 4, 2 and 5 for the first, second respectively the third game type. We have to buy the games. A game of the first, second resp. third type costs 5, 4 respectively 6 units of money.

We want to invite 50 guests and try to find and appropriate number of games of each type such that each guest can play in one of the games and we do not need to invest more that 60 units of money to buy the games.
The problem can be formalized as

$$
\begin{aligned}
4g_1 + 2g_2 + 5g_3 &= 50 && (1)\\
5g_1 + 4g_2 + 6g_3 &\leq 60 && (2)\\
g_1 &\geq 0 && (3)\\
g_2 &\geq 0 && (4)\\
g_3 &\geq 0 && (5)\\
& && (6)
\end{aligned}
$$

where g_1, g_2 resp. g_3 denote the number of games of the first, second resp. third type.

4.1 Eliminating the equations

We have one equation without any coefficient 1 or -1. Therefore we use our coefficient reduction technique to generate a new equation. The smallest coefficient is 2 thus we apply $\cdot \widehat{mod}\, 3$:

$$
\begin{aligned}
(4 \widehat{mod}\, 3)g_1 + (2 \widehat{mod}\, 3)g_2 + (5 \widehat{mod}\, 3)g_3 &= 3\sigma + (50 \widehat{mod}\, 3)\\
(4 - 3\lfloor 4/3 + 1/2 \rfloor)g_1 &\\
+(2 - 3\lfloor 2/3 + 1/2 \rfloor)g_2 &\\
+(5 - 3\lfloor 5/3 + 1/2 \rfloor)g_3 &= 3\sigma + (50 - 3\lfloor 50/3 + 1/2 \rfloor)\\
1 \cdot g_1 + (-1) \cdot g_2 + (-1) \cdot g_3 &= 3\sigma + (-1)\\
g_2 &= g_1 - g_3 - 3\sigma + 1
\end{aligned}
$$

We add this equation to our problem and use it to eliminate g_2, resulting in the problem

$$
\begin{aligned}
g_1 &\geq 0\\
g_1 - g_3 - 3\sigma &\geq -1\\
g_3 &\geq 0\\
g_3 &= -2g_1 + 2\sigma + 16\\
9g_1 + 2g_3 - 12\sigma &\leq 56
\end{aligned}
$$

Next we eliminate g_3 using the only equation. This gives us

$$\begin{aligned} g_1 &\geq 0 \\ 3g_1 - 5\sigma &\geq 15 \\ -g_1 + \sigma &\geq -8 \\ 5g_1 - 8\sigma &\leq 24 \end{aligned}$$

Now we have only inequations and we can start with the real shadow of the Omega test.

4.2 Real shadow

We first eliminate g_1. We have two lower bounds

$$\begin{aligned} 0 &\leq g_1 \\ 5\sigma + 15 &\leq 3g_1 \end{aligned}$$

and two upper bounds

$$\begin{aligned} g_1 &\leq \sigma + 8 \\ 5g_1 &\leq 8\sigma + 24 \end{aligned}$$

For the first lower and first upper bound we get

$$\begin{aligned} & 0 \leq \sigma + 8 \\ \Leftrightarrow\ & -8 \leq \sigma \end{aligned}$$

For the second lower and the first upper bound

$$\begin{aligned} & 5\sigma + 15 \leq 3\sigma + 24 \\ \Leftrightarrow\ & 2\sigma \leq 9 \\ \Leftrightarrow\ & \sigma \leq 4 \end{aligned}$$

For the first lower and the second upper bound

$$\begin{aligned} & 0 \leq 8\sigma + 24 \\ \Leftrightarrow\ & -24 \leq 8\sigma \\ \Leftrightarrow\ & -3 \leq \sigma \end{aligned}$$

Finally, for the second lower and second upper bound

$$\begin{aligned} & 25\sigma + 75 \leq 24\sigma + 72 \\ \Leftrightarrow\ & \sigma \leq -3 \end{aligned}$$

These constraints define the interval $[-3, -3]$ for σ which contains the integer -3, thus the real shadow returns without any conflict.

4.3 Dark shadow

Reminder: We can eliminate a variable x by introducing for each lower-upper-bound pair

$$\beta \leq bx, \quad cx \leq \gamma$$

a new constraint

$$b\gamma - c\beta \leq (c-1)(b-1).$$

Afterwards we can discard all constraints containing x.
We start eliminating the variable g_1, which is constrained by the two lower bounds

$$\begin{array}{rcl} 0 & \leq & g_1 \\ 5\sigma + 15 & \leq & 3g_1 \end{array}$$

and two upper bounds

$$\begin{array}{rcl} g_1 & \leq & \sigma + 8 \\ 5g_1 & \leq & 8\sigma + 24. \end{array}$$

For the first lower bound and the first upper bound we get

$$\begin{array}{lrcl} & 1 \cdot (\sigma + 8) - 1 \cdot 0 & \geq & (1-1)(1-1) \\ \Leftrightarrow & \sigma + 8 & \geq & 0 \\ \Leftrightarrow & \sigma & \geq & -8. \end{array}$$

For the second lower bound and the second upper bound we get

$$\begin{array}{lrcl} & 3 \cdot (8\sigma + 24) - 5 \cdot (5\sigma + 15) & \geq & (5-1)(3-1) \\ \Leftrightarrow & 24\sigma + 72 - 25\sigma - 75 & \geq & 8 \\ \Leftrightarrow & -\sigma & \geq & 11 \\ \Leftrightarrow & \sigma & \leq & -11. \end{array}$$

We get the conflict $-8 \leq \sigma \leq -11$ and know that the dark shadow contains no integer solution for g_1. Hence, we continue with the gray shadow.

4.4 Gray shadow

Reminder: For a lower-upper-bound pair $\beta \leq bx$ and $cx \leq \gamma$ try all possible values i with $bx = \beta + i$, $0 \leq i \leq \frac{cb-c-b}{c}$. For each of these values we extend the constraint set with the equation $bx = \beta + i$ and check it for satisfiability.

We first eliminate g_1. The lower bounds on g_1 are

$$\begin{aligned} 0 &\leq g_1 \\ 5\sigma + 15 &\leq 3g_1 \end{aligned}$$

and the upper bounds on g_1 are

$$\begin{aligned} g_1 &\leq \sigma + 8 \\ 5g_1 &\leq 8\sigma + 24. \end{aligned}$$

For the first lower bound and the first upper bound we get

$$g_1 = i, \quad 0 \leq i \leq -\frac{1}{5}$$

resulting in an empty set of cases.
For the first lower bound and the second upper bound we get

$$3g_1 = 5\sigma + 15 + i, \quad 0 \leq i \leq \left\lfloor \frac{5 \cdot 3 - 5 - 3}{5} \right\rfloor = 1$$

resulting in the following cases:

i) $3g_1 = 5\sigma + 15$

ii) $3g_1 = 5\sigma + 16$

Considering the first case we extend the system to

$$\begin{aligned} g_1 &\geq 0 \\ 3g_1 - 5\sigma &\geq 15 \\ -g_1 + \sigma &\geq -8 \\ 5g_1 - 8\sigma &\leq 24 \\ 3g_1 - 5\sigma &= 15 \end{aligned}$$

We use the equation $3g_1 = 5\sigma + 15$ to eliminate g_1. The smallest coefficient occuring is 3, therefore we apply $(\cdot \ \widehat{mod}\ 4)$:

$$\begin{aligned} & (3 \ \widehat{mod}\ 4)g_1 - (5 \ \widehat{mod}\ 4)\sigma &=& \ (15 \ \widehat{mod}\ 4) + 4\sigma_2 \\ \Leftrightarrow \quad & (-1)g_1 - (1)\sigma &=& \ (-1) + 4\sigma_2 \\ \Leftrightarrow \quad & g_1 &=& \ -\sigma - 4\sigma_2 + 1 \end{aligned}$$

and substitute g_1 by $-\sigma - 4\sigma_2 + 1$ in all constraints.

1. In $g_1 \geq 0$:
$$\begin{array}{rrcl} & -\sigma - 4\sigma_2 + 1 & \geq & 0 \\ \Leftrightarrow & \sigma + 4\sigma_2 & \leq & 1 \end{array}$$

2. In $3g_1 - 5\sigma \geq 15$:
$$\begin{array}{rrcl} & 3(-\sigma - 4\sigma_2 + 1) - 5\sigma & \geq & 15 \\ & -8\sigma - 12\sigma_2 & \geq & 12 \\ \Leftrightarrow & 2\sigma + 3\sigma_2 & \leq & -3 \end{array}$$

3. In $-g_1 + \sigma \geq -8$:
$$\begin{array}{rcl} -(-\sigma - 4\sigma_2 + 1) + \sigma & \geq & -8 \\ 2\sigma + 4\sigma_2 & \geq & -7 \end{array}$$

4. In $5g_1 - 8\sigma \leq 24$:
$$\begin{array}{rcl} 5(-\sigma - 4\sigma_2 + 1) - 8\sigma & \leq & 24 \\ -13\sigma - 20\sigma_2 & \leq & 19 \\ 13\sigma + 20\sigma_2 & \geq & -19 \end{array}$$

5. In $3g_1 - 5\sigma = 15$:
$$\begin{array}{rcl} 3(-\sigma - 4\sigma_2 + 1) & = & 5\sigma + 15 \\ -8\sigma - 12\sigma_2 & = & 12 \\ 2\sigma + 3\sigma_2 & = & -3 \end{array}$$

Thus, we still have a equation, but with a smaller greates coefficient than the equation we have just used to eliminate g_1. We continue with the new equation $2\sigma + 3\sigma_2 = -3$ to eliminate the variable with the smallest coefficient, i.e. σ. Note, that we eventually get rid of the equation. The smallest coefficient occuring now is 2, therefore we apply $(\cdot \ \widehat{mod}\ 3)$:

$$\begin{array}{rrcl} & (-2 \ \widehat{mod}\ 3)\sigma - (3 \ \widehat{mod}\ 3)\sigma & = & (3 \ \widehat{mod}\ 3) + 3\sigma_3 \\ \Leftrightarrow & (1)g_1 - (0)\sigma & = & (0) + 3\sigma_3 \\ \Leftrightarrow & \sigma & = & 3\sigma_3 \end{array}$$

and substitute σ by $3\sigma_3$ in all constraints.

1. In $\sigma + 4\sigma_2 \leq 1$:
$$\begin{array}{rrcl} & -3\sigma_3 - 4\sigma_2 + 1 & \geq & 0 \\ \Leftrightarrow & 3\sigma_3 + 4\sigma_2 & \leq & 1 \end{array}$$
2. In $2\sigma + 3\sigma_2 \leq -3$:
$$\begin{array}{rrcl} & 6\sigma_3 + 3\sigma_2 & \leq & -3 \\ \Leftrightarrow & 2\sigma_3 + \sigma_2 & \leq & -1 \end{array}$$
3. In $2\sigma + 4\sigma_2 \geq -7$:
$$6\sigma_3 + 4\sigma_2 \quad \geq \quad -7$$
4. In $13\sigma + 20\sigma_2 \geq -19$:
$$39\sigma_3 + 20\sigma_2 \quad \geq \quad -19$$
5. In $2\sigma + 3\sigma_2 = -3$:
$$\begin{array}{rrcl} & 6\sigma_3 + 3\sigma_2 & = & -3 \\ \Leftrightarrow & \sigma_2 & = & -2\sigma_3 - 1 \end{array}$$

Now, we eliminate σ_2 using $\sigma_2 = -2\sigma_3 - 1$ and get

1. In $3\sigma_3 + 4\sigma_2 \leq 1$:
$$\begin{array}{rrcl} & 3\sigma_3 - 8\sigma_2 & \leq & 5 \\ \Leftrightarrow & \sigma_3 & \geq & -1 \end{array}$$
2. In $2\sigma_3 + \sigma_2 \leq -1$:
$$\begin{array}{rrcl} & -2\sigma_3 + 2\sigma_3 + 1 & \geq & 1 \\ \Leftrightarrow & 1 & \geq & 1 \end{array}$$
3. In $6\sigma_3 + 4\sigma_2 \geq -7$:
$$\begin{array}{rrcl} & 6\sigma_3 + 4(-2\sigma_3 - 1) & \geq & -7 \\ \Leftrightarrow & -2\sigma_3 & \geq & -3 \\ \Leftrightarrow & \sigma_3 & \leq & 1 \end{array}$$
4. In $39\sigma_3 + 20\sigma_2 \geq -19$:
$$\begin{array}{rrcl} & 39\sigma_3 + 20(-2\sigma_3 - 1) & \geq & -19 \\ \Leftrightarrow & -\sigma_3 & \geq & 1 \\ \Leftrightarrow & \sigma_3 & \leq & -1 \end{array}$$

5. In $2\sigma_3 + \sigma_2 = -1$:

$$\begin{array}{rrcl} & 2\sigma_3 - 2\sigma_3 - 1 & = & -1 \\ \Leftrightarrow & -1 & = & -1 \end{array}$$

So $\sigma_3 = -1$ is a solution. Using the equations we can find a satisfying assignment as follows. From $\sigma_2 = -2\sigma_3 - 1$ we get $\sigma_2 = 1$, from $\sigma = 3\sigma_3$ we get $\sigma = -3$, from $g_1 = -\sigma - 4\sigma_2 + 1$ we get

$$g_1 = 0,$$

from $g_3 = -2g_1 + 2\sigma + 16$ we get

$$g_3 = 10,$$

and from $g_2 = g_1 - g_3 - 3\sigma + 1$ we get

$$g_2 = 0.$$

Considering the original problem

$$\begin{array}{rcl} 4g_1 + 2g_2 + 5g_3 & = & 50 \\ 5g_1 + 4g_2 + 6g_3 & \leq & 60 \\ g_1 & \geq & 0 \\ g_2 & \geq & 0 \\ g_3 & \geq & 0 \end{array}$$

you can see that it is indeed a satisfying assignment.