Satisfiability Checking
Gomory cuts (Sec. 5.3.1

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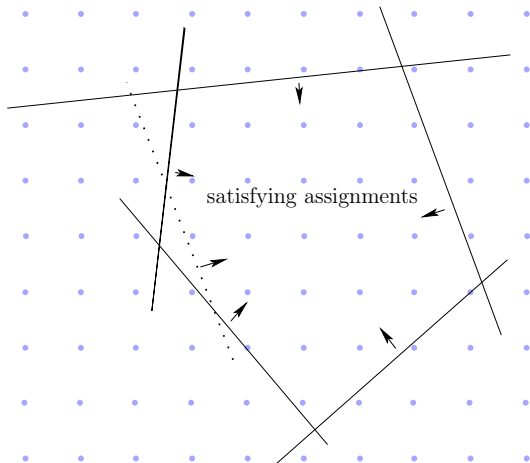
WS 14/15

Branch and bound

We use Simplex to find a real solution. If the solution is not integer-valued, we generate a new constraint such that the new (reduced) feasible region has two important properties:

- It does not contain the found non-integer solution any more.
- It still contains all feasible solutions to the original ILP problem.

Branch and bound

- We looked at branching by dividing the value domain of an integer variable into two halfs (branching).
- We could also cut with other, better contraints.
- E.g., for $x \in \mathbb{Z}$, from $2x \leq 11$ we can conclude $x \leq 5$.
- But how to generate such cutting planes?
- We look at one method for generating cutting planes: Gomory cuts.

Cutting planes, geometrically

satisfying assignments

The dotted line is a cutting plane.

Example: Gomory cuts

Suppose our input integer linear problem has
- integer variables x_1, x_2, x_3 and
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After solving the relaxed problem:

- The final tableau of the general Simplex algorithm includes the constraint

  $$x_3 = 0.5x_1 + 2.5x_2$$

- and the solution α is

  $$\{x_3 \mapsto 1.75, x_1 \mapsto 1, x_2 \mapsto 0.5\}$$

  with $1.75 = 0.5 \cdot 1 + 2.5 \cdot 0.5$.

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    with $1.75 = 0.5 \cdot 1 + 2.5 \cdot 0.5$.
- Subtracting these values from the variables gives us

    $$x_3 - 1.75 = 0.5(x_1 - 1) + 2.5(x_2 - 0.5) \ .$$

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- On the other hand, it is implied by the integer system of constraints, and hence cannot remove any *integer* solution.

Gomory cuts

- Generalizing this example:
    - Upper bounds.
    - Both positive and negative coefficients.
- The description that follows is based on
    - *Integrating Simplex with DPLL(T)*
      Technical report SRI-CSL-06-01
      Dutertre and de Moura (2006).

Gomory cuts

There are two preliminary conditions for deriving a Gomory cut from a constraint:

- The assignment to at least one basic or original variable is fractional.
- The non-basic variables are either additional variables or their coefficients are integers.
- One more constraint which we discuss later.

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  with $x_i \in \mathcal{B}$, $\alpha(x_i)$ not an integer and a_{ij} integer for all $j \in \mathcal{N}_o$.

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$$\underbrace{x_i - \sum_{x_j \in \mathcal{N}_o} a_{ij} x_j}_{T} = \sum_{x_j \in \mathcal{N}_a} a_{ij} x_j$$

Note: T is integer-valued.

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Assumption: $\alpha(T)$ is not an integer.

Gomory cuts

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- Then also

$$T - \alpha(T) = \sum_{j \in \mathcal{N}_a} a_{ij}(x_j - \alpha(x_j))$$

$$T - \lfloor \alpha(T) \rfloor = \underbrace{(\alpha(T) - \lfloor \alpha(T) \rfloor)}_{f_T} + \sum_{j \in \mathcal{N}_a} a_{ij}(x_j - \alpha(x_j))$$

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- It follows that
$$f_T + \sum_{j \in \mathcal{N}_a} a_{ij}(x_j - \alpha(x_j))$$
should be integer-valued. Note: $0 < f_T < 1$.

Gomory cuts

- Partition the non-basic additional variables to
    - those that are currently assigned their lower bound, and
    - those that are currently assigned their upper bound:

$$
\begin{aligned}
L &= \{j \mid x_j \in \mathcal{N}_a \wedge \alpha(x_j) = l_j\} \\
U &= \{j \mid x_j \in \mathcal{N}_a \wedge \alpha(x_j) = u_j\} \ .
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- We further split L and U as follows:

$$
\begin{aligned}
L^+ &= \{j \mid j \in L \wedge a_{ij} > 0\} \\
L^- &= \{j \mid j \in L \wedge a_{ij} < 0\} \\
U^+ &= \{j \mid j \in U \wedge a_{ij} > 0\} \\
U^- &= \{j \mid j \in U \wedge a_{ij} < 0\}
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- and further equals
$$f_T \quad + \quad \sum_{j \in L^+} a_{ij}(x_j - l_j) + \sum_{j \in L^-} a_{ij}(x_j - l_j)$$
$$- \quad \sum_{j \in U^-} a_{ij}(u_j - x_j) - \sum_{j \in U^+} a_{ij}(u_j - x_j)$$

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positive and integer-valued, thus
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- Gathering the positive components,
$$\sum_{j \in L^+} a_{ij}(x_j - l_j) - \sum_{j \in U^-} a_{ij}(u_j - x_j) \geq 1 - f_T ,$$
or, equivalently,
$$\sum_{j \in L^+} \frac{a_{ij}}{1 - f_T}(x_j - l_j) - \sum_{j \in U^-} \frac{a_{ij}}{1 - f_T}(u_j - x_j) \geq 1 .$$

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- Gathering the negative components,
$$\sum_{j \in L^-} a_{ij}(x_j - l_j) - \sum_{j \in U^+} a_{ij}(u_j - x_j) \leq -f_T \ .$$

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$$\sum_{j \in L^-} a_{ij}(x_j - l_j) - \sum_{j \in U^+} a_{ij}(u_j - x_j) \leq -f_T .$$

- Dividing by $-f_T$ gives us
$$-\sum_{j \in L^-} \frac{a_{ij}}{f_T}(x_j - l_j) + \sum_{j \in U^+} \frac{a_{ij}}{f_T}(u_j - x_j) \geq 1 .$$

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- Case 1: $\sum_{j \in L} a_{ij}(x_j - l_j) - \sum_{j \in U} a_{ij}(u_j - x_j) > 0$:

$$\sum_{j \in L^+} \frac{a_{ij}}{1 - f_T}(x_j - l_j) - \sum_{j \in U^-} \frac{a_{ij}}{1 - f_T}(u_j - x_j) \geq 1 .$$

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- Case 2: $\sum_{j \in L} a_{ij}(x_j - l_j) - \sum_{j \in U} a_{ij}(u_j - x_j) \leq 0$
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- Therefore these two equations imply (note that all sums-blocks are non-negative)
$$\sum_{j \in L^+} \frac{a_{ij}}{1 - f_T}(x_j - l_j) - \sum_{j \in L^-} \frac{a_{ij}}{f_T}(x_j - l_j)$$
$$+ \sum_{j \in U^+} \frac{a_{ij}}{f_T}(u_j - x_j) - \sum_{j \in U^-} \frac{a_{ij}}{1 - f_T}(u_j - x_j) \geq 1 \, .$$

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$$\sum_{j \in L^+} \frac{a_{ij}}{1 - f_T}(x_j - l_j) - \sum_{j \in L^-} \frac{a_{ij}}{f_T}(x_j - l_j)$$
$$+ \sum_{j \in U^+} \frac{a_{ij}}{f_T}(u_j - x_j) - \sum_{j \in U^-} \frac{a_{ij}}{1 - f_T}(u_j - x_j) \;\; \geq 1 \,.$$

- Since each of the elements on the left-hand side is equal to zero under the current assignment α, this assignment α is ruled out by the new constraint.

- In other words: the solution to the linear problem augmented with the constraint is guaranteed to be different from the previous one.