Satisfiability Checking
The Simplex Algorithm

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Outline

Gaussian elimination

- Given a linear system $Ax = b$

$$
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1k} \\
a_{21} & a_{22} & \ldots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{k1} & a_{k2} & \ldots & a_{kk}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_k
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_k
\end{pmatrix}
$$

- Manipulate $A|b$ to obtain an upper-triangular form

$$
\left(
\begin{array}{cccc|c}
a'_{11} & a'_{12} & \ldots & a'_{1k} & b'_1 \\
0 & a'_{22} & \ldots & a'_{2k} & b'_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & a'_{kk} & b'_k
\end{array}
\right)
$$

Gaussian elimination

Then, solve backwards from k's row according to:

$$x_i = \frac{1}{a'_{ii}}(b'_i - \sum_{j=i+1}^{k} a'_{ij}x_j)$$

Example

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & 4 \\ 4 & -1 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ -2 & 3 & 4 & 3 \\ 4 & -1 & -8 & 9 \end{array} \right)$$

$$
\begin{array}{rclc}
R3 & = & (& 4, & -1, & -8 & | & 9 &) \\
-4R1 & = & (& -4, & -8, & -4 & | & -24 &) \\
R3 & += & & -4R1
\end{array}
\qquad
\left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ -2 & 3 & 4 & 3 \\ 0 & -9 & -12 & -15 \end{array} \right)
$$

$$
\begin{array}{rclc}
R2 & = & (& -2, & 3, & 4 & | & 3 &) \\
2R1 & = & (& 2, & 4, & 2 & | & 12 &) \\
R2 & += & & 2R1
\end{array}
\qquad
\left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 7 & 6 & 15 \\ 0 & -9 & -12 & -15 \end{array} \right)
$$

$$
\begin{array}{rclc}
R3 & = & (& 0, & -9, & -12 & | & -15 &) \\
\frac{9}{7}R2 & = & (& 0, & 9, & \frac{6\cdot9}{7} & | & \frac{15\cdot9}{7} &) \\
R3 & += & & \frac{9}{7}R2
\end{array}
\qquad
\left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 7 & 6 & 15 \\ 0 & 0 & -\frac{30}{7} & \frac{30}{7} \end{array} \right)
$$

Now: $x_3 = -1$, $x_2 = 3$, $x_1 = 1$. Problem solved!

Satisfiability with Simplex

- Simplex was originally designed for solving the optimization problem:

  max $\vec{c}\,\vec{x}$
  s.t.
  $A\vec{x} \leq \vec{b}, \quad \vec{x} \geq \vec{0}$

- We are only interested in the feasibility problem
  ($=$ satisfiability problem).

General Simplex

- We will learn a variant called general simplex.
- Well-suited for solving the satisfiability problem fast.

- The input: $A\vec{x} \leq \vec{b}$

  - A is a $m \times n$ coefficient matrix
  - The problem variables are $\vec{x} = x_1, \ldots, x_n$

- First step: convert the input to *general form*

General form

Definition (General Form)

$$A(\vec{x}, \vec{s}) = 0 \quad \text{and} \quad \bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i$$

A combination of

- Linear equalities of the form $\sum_i a_i x_i = 0$
- Lower and upper bounds on variables

Transformation to general form

- Replace $\sum_i a_i x_i \bowtie b_j$ (where $\bowtie \in \{=, \leq, \geq\}$)

  with $\sum_i a_i x_i - s_j = 0$

  and $s_j \bowtie b_j$.

- Note: no $>, <$!

- $s_1, \ldots, s_m$ are called the *additional variables*

Example 1

Convert $x + y \geq 2$!

Result:
$x + y - s_1 = 0$
$s_1 \geq 2$

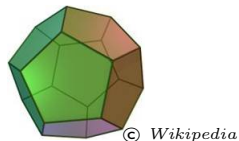It is common to keep the conjunctions implicit

Example 2

Convert

$$
\begin{aligned}
x \quad +y \quad &\geq 2 \\
2x \quad -y \quad &\geq 0 \\
-x \quad +2y \quad &\geq 1
\end{aligned}
$$

Result:

$$
\begin{aligned}
x \quad +y \quad -s_1 \quad &= 0 \\
2x \quad -y \quad -s_2 \quad &= 0 \\
-x \quad +2y \quad -s_3 \quad &= 0 \\
s_1 \quad &\geq 2 \\
s_2 \quad &\geq 0 \\
s_3 \quad &\geq 1
\end{aligned}
$$

Linear inequality constraints,
geometrically, define a
convex polyhedron.

© *Wikipedia*

Our example from before:

$$\begin{aligned} x &+y &\geq 2 \\ 2x &-y &\geq 0 \\ -x &+2y &\geq 0 \end{aligned}$$

Matrix form

- Recall the general form: $A(\vec{x}, \vec{s}) = 0$ and $\bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i$

- A is now an $m \times (n + m)$ matrix
  due to the additional variables.

$$
\begin{array}{rrrl}
x & +y & -s_1 & = 0 \\
2x & -y & -s_2 & = 0 \\
-x & +2y & -s_3 & = 0 \\
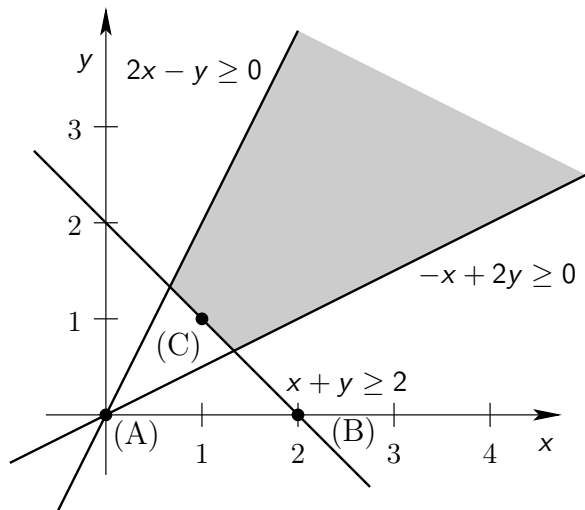& & s_1 & \geq 2 \\
& & s_2 & \geq 0 \\
& & s_3 & \geq 1
\end{array}
\qquad
\begin{array}{ccccc}
x & y & s_1 & s_2 & s_3 \\
\left(\begin{array}{ccccc}
1 & 1 & -1 & 0 & 0 \\
2 & -1 & 0 & -1 & 0 \\
-1 & 2 & 0 & 0 & -1
\end{array} \right)
\end{array}
$$

non-basic

The tableau

- The diagonal part is inherent to the general form:

$$
\begin{array}{ccccc}
x & y & s_1 & s_2 & s_3 \\
\end{array}
$$
$$
\left(
\begin{array}{ccccc}
1 & 1 & -1 & 0 & 0 \\
2 & -1 & 0 & -1 & 0 \\
-1 & 2 & 0 & 0 & -1 \\
\end{array}
\right)
$$

- Instead, we can write:

$$
\begin{array}{c}
\\
s_1 \\
s_2 \\
s_3 \\
\end{array}
\begin{array}{cc}
x & y \\
\end{array}
\left(
\begin{array}{cc}
1 & 1 \\
2 & -1 \\
-1 & 2 \\
\end{array}
\right)
$$

The tableau

- The tableaux changes throughout the algorithm, but maintains its $m \times n$ structure
- Distinguish basic (also called dependent) and non-basic variables

$$
\text{Basic variables} \rightarrow
\begin{matrix}
& x & y \\
s_1 & & \\
s_2 & & \\
s_3 & &
\end{matrix}
\left(
\begin{matrix}
1 & 1 \\
2 & -1 \\
-1 & 2
\end{matrix}
\right)
\quad \longleftarrow \text{Non-basic variables}
$$

Notation:

$\mathcal{B}$ the set of basic variables

$\mathcal{N}$ the set of non-basic variables

- Initially, basic variables = the additional variables
- The tableaux is simply a different notation for the system

$$
\bigwedge_{s_i \in \mathcal{B}} \left(s_i = \sum_{x_j \in \mathcal{N}} a_{ij} x_j \right)
$$

Data structures

- Simplex maintains:
    - The tableau,
    - an assignment α to all (problem and additional) variables.
- Initially, $\alpha(x_i) = 0$ for $i \in \{1, ..., n + m\}$

- Two invariants are maintained throughout:
    1. $A\vec{x} = 0$
    2. All non-basic variables satisfy their bounds
- The basic variables do not need to satisfy their bounds.

- Can you see why these invariants are maintained initially?

Invariants

- The initial assignment satisfies $A\vec{x} = 0$

- If the bounds of all basic variables are satisfied by α, return "satisfiable".

- Otherwise... *pivot*.

Pivoting

1 Find a basic variable x_i that violates its bounds.
Suppose that $\alpha(x_i) < l_i$.

2 Find a non-basic variable x_j such that
- $a_{ij} > 0$ and $\alpha(x_j) < u_j$, or
- $a_{ij} < 0$ and $\alpha(x_j) > l_j$.

Why? Such a variable is called suitable.

3 If there is no suitable variable, return "unsatisfiable".

Why?

1. Solve equation i for x_j:

$$\text{From:} \quad x_i = a_{ij}x_j + \sum_{k \neq j} a_{ik}x_k$$

$$\text{To:} \quad x_j = \frac{x_i}{a_{ij}} - \sum_{k \neq j} \frac{a_{ik}}{a_{ij}}x_k$$

2. Swap x_i and x_j, and update the i-th row accordingly

From:

a_{i1}	$\ldots$	a_{ij}	$\ldots$	a_{in}

To:

$\dfrac{-a_{i1}}{a_{ij}}$	$\ldots$	$\dfrac{1}{a_{ij}}$	$\ldots$	$\dfrac{-a_{in}}{a_{ij}}$

3 Update all other rows:
Replace x_j with its equivalent obtained from row i:

$$x_j = \frac{x_i}{a_{ij}} - \sum_{k \neq j} \frac{a_{ik}}{a_{ij}} x_k$$

4 Update α as follows:

- Increase $\alpha(x_j)$ by $\theta = \frac{l_i - \alpha(x_i)}{a_{ij}}$
  Now x_j is a basic variable: it may violate its bounds.
  Update $\alpha(x_i)$ accordingly.
  Q: What is $\alpha(x_i)$ now?

- Update α for all other basic (dependent) variables.

Pivoting: Example (1)

- Recall the tableau and constraints in our example:

	x	y
s_1	1	1
s_2	2	-1
s_3	-1	2

$$
\begin{aligned}
2 &\leq s_1 \\
0 &\leq s_2 \\
1 &\leq s_3
\end{aligned}
$$

- Initially, α assigns 0 to all variables

  $\implies$ Violated are the bounds of s_1 and s_3

- We will fix s_1.

- x is a *suitable* non-basic variable for pivoting.
  It has no upper bound!

- So now we pivot s_1 with x

Pivoting: Example (2)

	x	y
s_1	1	1
s_2	2	-1
s_3	-1	2

$$2 \leq s_1$$
$$0 \leq s_2$$
$$1 \leq s_3$$

- Solve 1st row for x:

$$s_1 = x + y \quad \leftrightarrow \quad x = s_1 - y$$

- Replace x in other rows:

$$s_2 = 2(s_1 - y) - y \quad \leftrightarrow \quad s_2 = 2s_1 - 3y$$
$$s_3 = -(s_1 - y) + 2y \quad \leftrightarrow \quad s_3 = -s_1 + 3y$$

Pivoting: Example (3)

This results in the following new tableau:

$$
\begin{aligned}
x &= s_1 - y \\
s_2 &= 2s_1 - 3y \\
s_3 &= -s_1 + 3y
\end{aligned}
$$

	s_1	y
x	1	-1
s_2	2	-3
s_3	-1	3

$$
\begin{aligned}
2 &\leq s_1 \\
0 &\leq s_2 \\
1 &\leq s_3
\end{aligned}
$$

What about the assignment?

- We should increase x by $\theta = \dfrac{2 - 0}{1} = 2$
- Hence, $\alpha(x) = 0 + 2 = 2$
- Now s_1 is equal to its lower bound: $\alpha(s_1) = 2$
- Update all the others

The new state:

	s_1	y
x	1	-1
s_2	2	-3
s_3	-1	3

$$
\begin{aligned}
\alpha(x) &= 2 \\
\alpha(y) &= 0 & 2 &\leq s_1 \\
\alpha(s_1) &= 2 & 0 &\leq s_2 \\
\alpha(s_2) &= 4 & 1 &\leq s_3 \\
\alpha(s_3) &= -2
\end{aligned}
$$

- Now s_3 violates its lower bound
- Which non-basic variable is suitable for pivoting?
  That's right... y
- We should increase y by $\theta = \frac{1-(-2)}{3} = 1$

The final state:

	s_1	s_3
x	$2/3$	$-1/3$
s_2	1	-1
y	$1/3$	$1/3$

$$\alpha(x) = 1$$
$$\alpha(y) = 1$$
$$\alpha(s_1) = 2$$
$$\alpha(s_2) = 1$$
$$\alpha(s_3) = 1$$

$$2 \leq s_1$$
$$0 \leq s_2$$
$$1 \leq s_3$$

All constraints are satisfied.

Observations I

The additional variables:

- Only additional variables have bounds.

- These bounds are permanent.

- Additional variables enter the base only on extreme points (their lower or upper bounds).

- When entering the base, they shift towards the other bound and possibly cross it (violate it).

Q: Can it be that we pivot x_i, x_j and then pivot x_j, x_i
   and thus enter a (local) cycle?

A: No.

- For example, suppose that $a_{ij} > 0$.
- We increased $\alpha(x_j)$ so now $\alpha(x_i) = l_i$.
- After pivoting, possibly $\alpha(x_j) > u_j$, but $a'_{ij} = 1/a_{ij} > 0$,
  hence the coefficient of x_i is not suitable

Termination

Is termination guaranteed?

- Not obvious. Perhaps there are bigger cycles.

- In order to avoid circles, we use Bland's rule:
    1. Determine a total order on the variables
    2. Choose the first basic variable that violates its bounds, and the first non-basic suitable variable for pivoting.
    3. It can be shown that this guarantees that no base is repeated, which implies termination.

General simplex with Bland's rule

1 Transform the system into the general form

$$A(\vec{x}, \vec{s}) = 0 \quad \text{and} \quad \bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i \ .$$

2 Set $\mathcal{B}$ to be the set of additional variables $s_1, \ldots, s_m$.

3 Construct the tableau for A.

4 Determine a fixed order on the variables.

5 If there is no basic variable that violates its bounds, return "satisfiable". Otherwise, let x_i be the first basic variable in the order that violates its bounds.

6 Search for the first suitable non-basic variable x_j in the order for pivoting with x_i. If there is no such variable, return "unsatisfiable".

7 Perform the pivot operation on x_i and x_j.

8 Go to step 5.