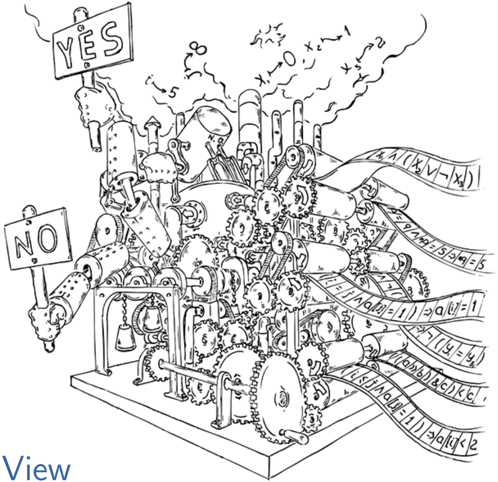
Bit-Vectors

Chapter 6



Decision Procedures

An Algorithmic Point of View

D.Kroening O.Strichman

Revision 1.2

Outline

What kind of logic do we need for system-level software?

```
State { int created = 0; }

IoCreateDevice.exit {
   if ($return==STATUS_SUCCESS)
     created = 1;
}

IoDeleteDevice.exit { created = 0; }

fun_AddDevice.exit {
   if (created && (pdevobj->Flags & DO_DEVICE_INITIALIZING) != 0) {
     abort "AddDevice routine failed to set "
           "~DO_DEVICE_INITIALIZING flag";
   }
}
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An Invariant of Microsoft Windows Device Drivers

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Decision Procedures for System-Level Software

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Bit-wise AND

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- We need bit-vector logic – with bit-wise operators, arithmetic overflow
- We want to scale to large programs – must verify large formulas

What kind of logic do we need for system-level software?

- We need bit-vector logic – with bit-wise operators, arithmetic overflow
- We want to scale to large programs – must verify large formulas
- Examples of program analysis tools that generate bit-vector formulas:
  - CBMC
  - SATABS
  - F-Soft (NEC)
  - SATURN (Stanford, Alex Aiken)
  - EXE (Stanford, Dawson Engler, David Dill)
  - Variants of those developed at IBM, Microsoft

Bit-Vector Logic: Syntax

$$
\begin{array}{rcl}
\mathit{formula} & : & \mathit{formula} \vee \mathit{formula} \mid \neg \mathit{formula} \mid \mathit{atom} \\
\mathit{atom} & : & \mathit{term}\ \mathit{rel}\ \mathit{term} \mid \mathit{Boolean\text{-}Identifier} \mid \mathit{term}[\,\mathit{constant}\,] \\
\mathit{rel} & : & = \ \mid\ < \\
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$$

- $\sim x$: bit-wise negation of x
- $ext(x)$: sign- or zero-extension of x
- $x << d$: left shift with distance d
- $x \circ y$: concatenation of x and y

Danger!

$$(x - y > 0) \iff (x > y)$$

Valid over $\mathbb{R}/\mathbb{N}$, but not over the bit-vectors.
(Many compilers have this sort of bug)

Width and Encoding

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- Typical encodings:
  - Binary encoding

    $$\langle x \rangle_U := \sum_{i=0}^{l-1} a_i \cdot 2^i$$

  - Two's complement

    $$\langle x \rangle_S := -2^{n-1} \cdot a_{n-1} + \sum_{i=0}^{l-2} a_i \cdot 2^i$$

  - But maybe also fixed-point, floating-point, ...

Examples

$$
\begin{aligned}
\langle 11001000 \rangle_U &= 200 \\
\langle 11001000 \rangle_S &= -128 + 64 + 8 = -56 \\
\langle 01100100 \rangle_S &= 100
\end{aligned}
$$

Notation to clarify width and encoding:

$$x_{[32]S}$$

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Width in bits

U: unsigned binary
S: signed two's complement

Definition (Bit-Vector)

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The value of bit number i of x is $x(i)$.

$$\underbrace{\begin{array}{|c|c|c|c|c|c|}\hline b_{l-1} & b_{l-2} & \cdots\cdots & b_2 & b_1 & b_0 \\ \hline\end{array}}_{l \text{ bits}}$$

We also write x_i for $x(i)$.

Lambda-Notation for Bit-Vectors

λ expressions are functions without a name

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Examples:

- The vector of length l that consists of zeros:

$$\lambda i \in \{0, \ldots, l-1\}.0$$

- A function that inverts (flips all bits in) a bit-vector:

$$\textit{bv-invert}(x) := \lambda i \in \{0, \ldots, l-1\}.\neg x_i$$

- A bit-wise OR:

$$\textit{bv-or}(x, y) := \lambda i \in \{0, \ldots, l-1\}.(x_i \vee y_i)$$

$\Longrightarrow$ we now have semantics for the bit-wise operators.

Example

$$(x_{[10]} \circ y_{[5]})[14] \iff x[9]$$

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- Final result:

$$(\lambda i.(i < 5) ? y_i : x_{i-5})(14) \iff x_9$$

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unsigned char number = 200;
number = number + 100;
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$\Longrightarrow$ Bit-vector arithmetic uses modular arithmetic!

Semantics for addition, subtraction:

$$a_{[l]} +_U b_{[l]} = c_{[l]} \quad \Longleftrightarrow \quad \langle a \rangle_U + \langle b \rangle_U = \langle c \rangle_U \mod 2^l$$
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\end{aligned}
$$

We can even mix the encodings:

$$
a_{[l]U} +_U b_{[l]S} = c_{[l]U} \quad \iff \quad \langle a \rangle_U + \langle b \rangle_S = \langle c \rangle_U \mod 2^l
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Semantics for Relational Operators

Semantics for $<$, $\leq$, $\geq$, and so on:

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\begin{aligned}
a_{[l]U} < b_{[l]U} &\iff \langle a \rangle_U < \langle b \rangle_U \\
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Mixed encodings:

$$a_{[l]U} < b_{[l]S} \iff \langle a \rangle_U < \langle b \rangle_S$$
$$a_{[l]S} < b_{[l]U} \iff \langle a \rangle_S < \langle b \rangle_U$$

Note that most compilers don't support comparisons with mixed encodings.

Complexity

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- It is NP-complete otherwise.

A Simple Decision Procedure

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- Most commonly used decision procedure
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Bit-Vector Flattening

1. Convert propositional part as before
2. Add a *Boolean variable for each bit* of each sub-expression (term)
3. Add *constraint* for each sub-expression

We denote the new Boolean variable for bit i of term t by $\mu(t)_i$.

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- This is easy for the bit-wise operators.

- Example for $a|_{[l]}b$:

$$\bigwedge_{i=0}^{l-1}(\mu(t)_i = (a_i \vee b_i))$$

(read $x = y$ over bits as $x \iff y$)

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- We can transform this into CNF using Tseitin's method.

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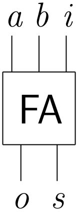
$\longrightarrow$ we can build a *circuit* that adds them!



Full Adder

$$
\begin{aligned}
s &\equiv (a + b + i) \bmod 2 &&\equiv a \oplus b \oplus i \\
o &\equiv (a + b + i) \operatorname{div} 2 &&\equiv a \cdot b + a \cdot i + b \cdot i
\end{aligned}
$$

The full adder in CNF:

$$
\begin{aligned}
&(a \vee b \vee \neg o) \wedge (a \vee \neg b \vee i \vee \neg o) \wedge (a \vee \neg b \vee \neg i \vee o) \wedge \\
&(\neg a \vee b \vee i \vee \neg o) \wedge (\neg a \vee b \vee \neg i \vee o) \wedge (\neg a \vee \neg b \vee o)
\end{aligned}
$$

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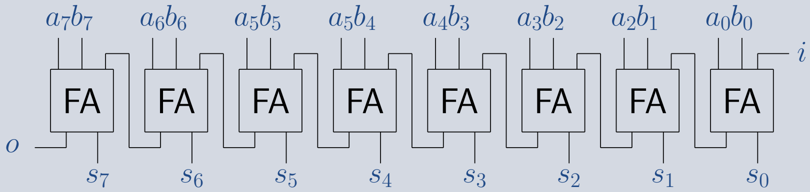
8-Bit ripple carry adder (RCA)



- Also called *carry chain adder*
- Adds l variables
- Adds $6 \cdot l$ clauses

- Multipliers result in very hard formulas
- Example:

$$a \cdot b = c \land b \cdot a \neq c \land x < y \land x > y$$

CNF: About 11000 variables, unsolvable for current SAT solvers

- Similar problems with division, modulo
- Q: Why is this hard?

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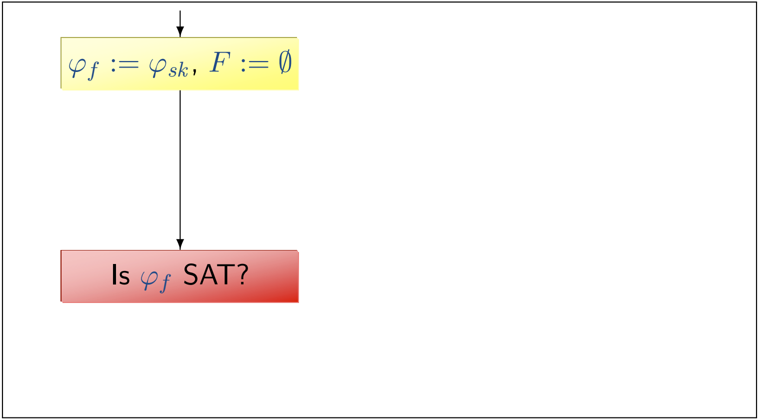
- Similar problems with division, modulo
- Q: Why is this hard?
- Q: How do we fix this?

Incremental Flattening

$\varphi_f := \varphi_{sk}$, $F := \emptyset$

φ_{sk}: Boolean part of φ
F: set of terms that are in the encoding



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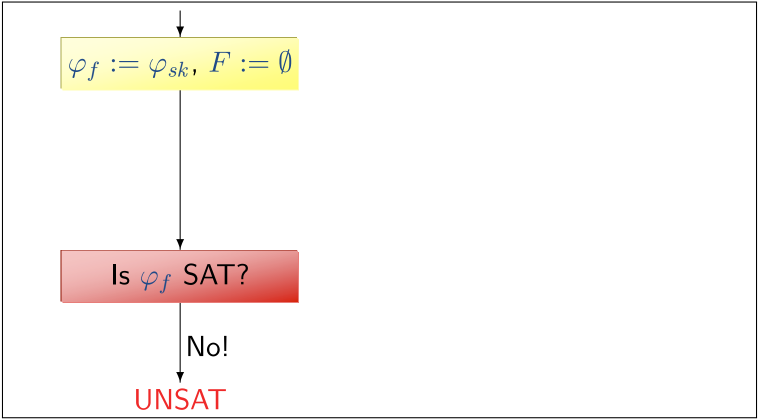

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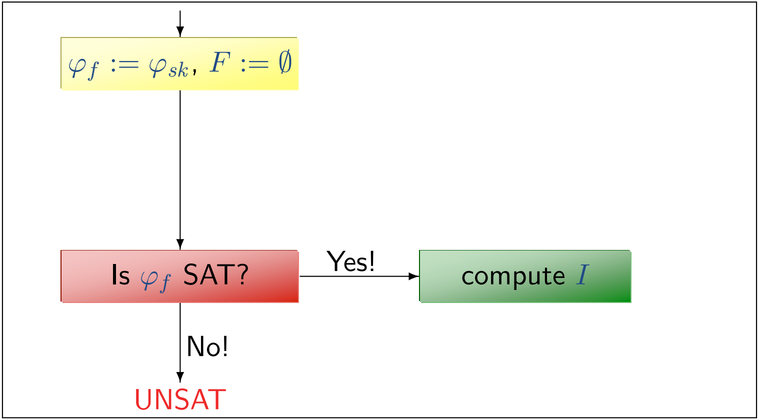

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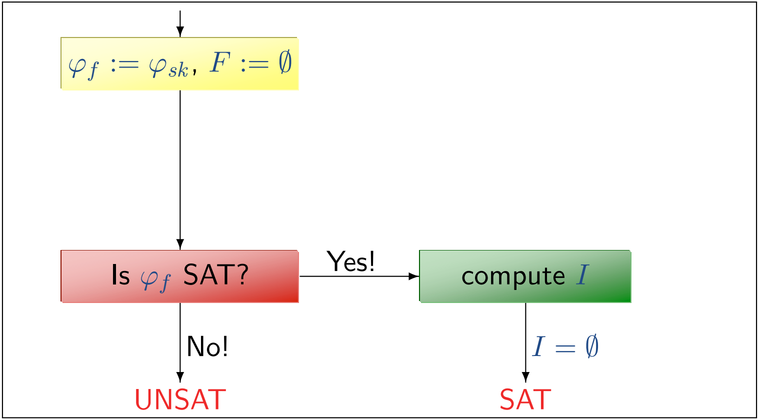

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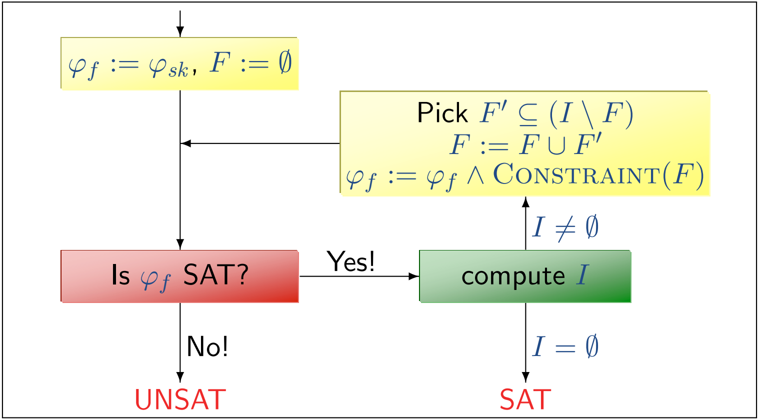

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Incremental Flattening

- Idea: add 'easy' parts of the formula first
- Only add hard parts when needed
- φ_f only gets stronger – use an incremental SAT solver

Incomplete Assignments

- Hey: initially, we only have the skeleton!
  How do we know what terms are inconsistent with the current assignment if the variables aren't even in φ_f?

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- Ideas:
  - All zeros
  - Sign extension for signed bit-vectors
  - Try to propagate constants ($a = b + 1$)