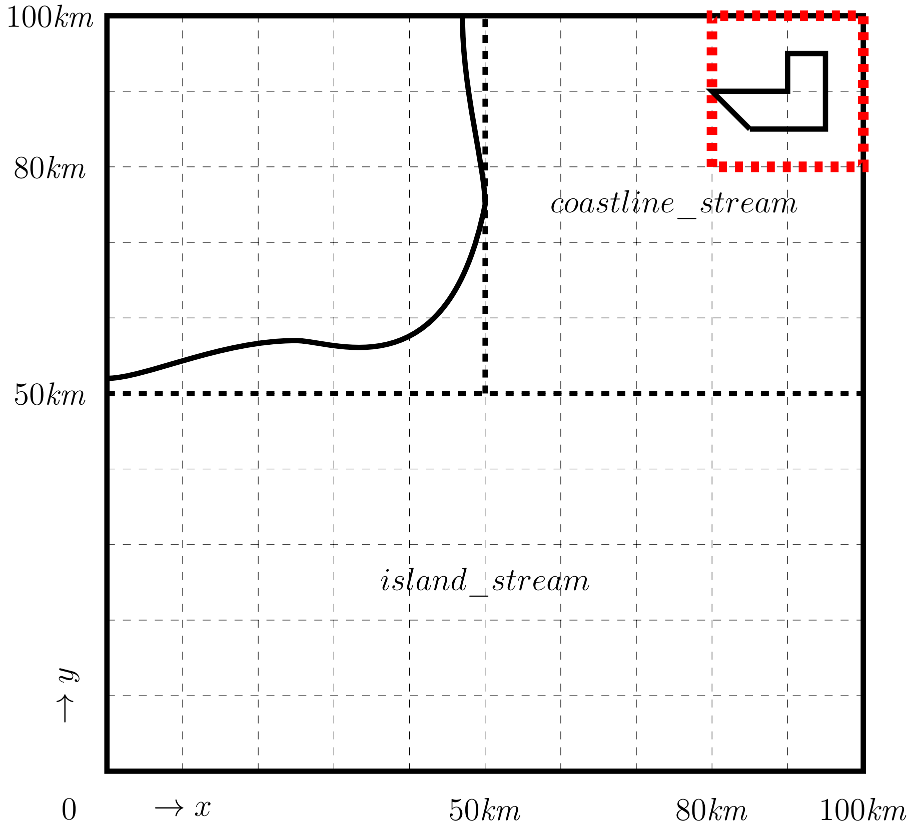
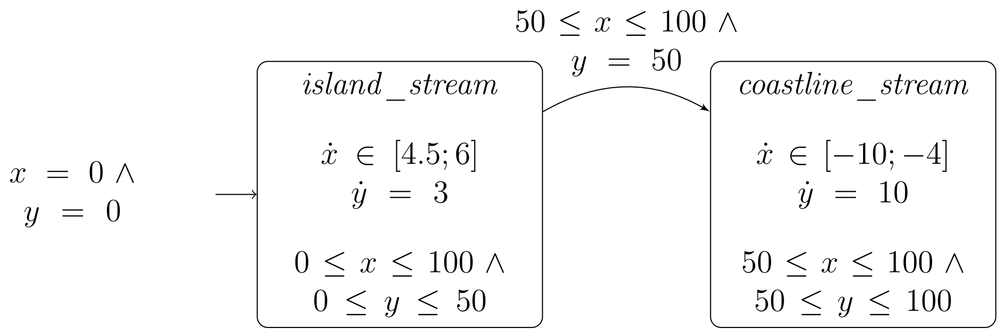
Modeling and Analysis of Hybrid Systems - SS 2015

Series 7

Exercise 1



A group of pupils is stranded on an island after their airplane crashed (cf. "The lord of the flies", William Golding 1954). They want to send a bottle message to get rescued. The strongest of them (Jack) throws the bottle into the stream (starting position of the bottle: $(0, 0)$) at the island, which is quite slow (modelled by location *island_stream*). Near the coastline, the stream speeds up and changes direction (location *coastline_stream*). If the bottle can reach the small area northeast (see drawing), where a military ship is located, the pupils have a chance to be rescued.



a) Please use the forward analysis algorithm presented in the lecture to check, whether there exists a possibility of a rescue (that means, determine whether the state $(\mathit{coastline\_stream}, 80 \leq x \leq 100 \wedge 80 \leq y \leq 100)$ can be reached).

b) Please sketch the reachable area in the map.

<u>Solution:</u>

a) We apply forward analysis, as presented in the lecture, using the following notations:

$$
\begin{aligned}
l_0 &:= \mathit{island\_stream}\\
l_1 &:= \mathit{coastline\_stream}\\
\texttt{Init} &:= \{(l_0, \underbrace{x = 0 \wedge y = 0}_{=:\texttt{Init}_{l_0}})\}\\
\texttt{Inv} &:= \{(l_0, \underbrace{0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50}_{=:\texttt{Inv}_{l_0}},\\
&\qquad (l_1, \underbrace{50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100}_{=:\texttt{Inv}_{l_1}})\}\\
\texttt{Act} &:= \{(l_0, \underbrace{x + 4.5t \leq x' \wedge x' \leq x + 6t \wedge y' = y + 3t}_{=:\texttt{Act}_{l_0}}),\\
&\qquad (l_1, \underbrace{x - 10t \leq x' \wedge x' \leq x - 4t \wedge y' = y + 10t}_{=:\texttt{Act}_{l_1}})\}\\
P^{bad} &:= \{(l_1, \underbrace{80 \leq x \wedge x \leq 100 \wedge 80 \leq y \wedge y \leq 100}_{=:\varphi^{bad}})\}
\end{aligned}
$$

Furthermore we use e to denote the only transition in the system:

$$
e := (l_0, \underbrace{50 \leq x \wedge x \leq 100 \wedge y = 50}_{=:\varphi_e^{guard}}, \underbrace{x' = x \wedge y' = y}_{=:\varphi_e^{reset}}, l_1)\,.
$$

$$
\begin{aligned}
P^0 =& \mathcal{T}^+(\texttt{Init} \,\hat{\cap}\, \texttt{Inv}) \\
=& \mathcal{T}^+(\{(l_0, \underbrace{x = 0 \wedge y = 0}_{\texttt{Init}_{l_0}} \wedge \underbrace{0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50}_{\texttt{Inv}_{l_0}})\}) \\
=& \mathcal{T}^+(\{(l_0, \underbrace{x = 0 \wedge y = 0}_{=:\varphi^0_{l_0}})\}) \\
=& \{(l_0, \mathcal{T}^+_{l_0}(\underbrace{x = 0 \wedge y = 0}_{\varphi^0_{l_0}}))\} \\
=& \{(l_0, \exists t. \boxed{\exists x^{pre}}. \boxed{\exists y^{pre}}.\ 0 \leq t \wedge \underbrace{\boxed{x^{pre} = 0} \wedge \boxed{y^{pre} = 0}}_{\varphi^0_{l_0}[x^{pre}, y^{pre}/x, y]} \wedge \\
& \underbrace{x^{pre} + 4.5t \leq x \wedge x \leq x^{pre} + 6t \wedge y = y^{pre} + 3t}_{\texttt{Act}_{l_0}[x^{pre}, y^{pre}, x, y/x, y, x', y']} \wedge \\
& \underbrace{0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50}_{\texttt{Inv}_{l_0}} \\
\overset{elim.x^{pre}, y^{pre}}{=} & \{(l_0, \boxed{\exists t}.\ 0 \leq t \wedge 4.5t \leq x \wedge x \leq 6t \wedge \boxed{y = 3t} \wedge 0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50)\} \\
\overset{elim.t}{=} & \{(l_0, 0 \leq \frac{1}{3}y \wedge 4.5 \cdot \frac{1}{3}y \leq x \wedge x \leq 6 \cdot \frac{1}{3}y \wedge 0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50)\} \\
=& \{(l_0, 1.5y \leq x \wedge x \leq 2y \wedge 0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50)\} \\
P^0 \hat{\cap} P^{bad} =& \emptyset
\end{aligned}
$$

$$
\begin{aligned}
R :=& (l_0, \underbrace{1.5y \leq x \wedge x \leq 2y \wedge 0 \leq x \wedge x \leq 100 \wedge 0 \leq y \wedge y \leq 50}_{=:\varphi^R}) \\
R' :=& (l_1, D^+_e(\varphi^R)) \\
=& (l_1, \boxed{\exists x^{pre}}. \boxed{\exists y^{pre}}. \\
& \underbrace{1.5y^{pre} \leq x^{pre} \wedge x^{pre} \leq 2y^{pre} \wedge 0 \leq x^{pre} \wedge x^{pre} \leq 100 \wedge 0 \leq y^{pre} \wedge y^{pre} \leq 50}_{\varphi^R[x^{pre}, y^{pre}/x, y]} \wedge \\
& \underbrace{50 \leq x^{pre} \wedge x^{pre} \leq 100 \wedge y^{pre} = 50}_{\varphi^{guard}_e[x^{pre}, y^{pre}/x, y]} \wedge \\
& \underbrace{\boxed{x = x^{pre}} \wedge \boxed{y = y^{pre}}}_{\varphi^{reset}_e[x^{pre}, y^{pre}, x, y/x, y, x', y']} \wedge \\
& \underbrace{50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100}_{\texttt{Inv}_{l_1}}) \\
\overset{elim.x^{pre}, y^{pre}}{=} & (l_1, \underbrace{75 \leq x \wedge x \leq 100 \wedge y = 50}_{=:\varphi^{R'}})
\end{aligned}
$$

$$
\begin{aligned}
R'' :=& (l_1, \mathcal{T}^{+}_{l_1}(\varphi^{R'})) \\
=& (l_1, \exists t.\exists x^{pre}.\boxed{\exists y^{pre}}.\ 0 \leq t \wedge \underbrace{75 \leq x^{pre} \wedge x^{pre} \leq 100 \wedge \boxed{y^{pre} = 50}}_{\varphi^{R'}[x^{pre}, y^{pre}/x,y]} \wedge \\
& \underbrace{x^{pre} - 10t \leq x \wedge x \leq x^{pre} - 4t \wedge y = y^{pre} + 10t}_{\texttt{Act}_{l_1}} \wedge \\
& \underbrace{50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100}_{\texttt{Inv}_{l_1}}) \\
\overset{elim.y^{pre}}{=}& (l_1, \exists t.\exists x^{pre}.\ 0 \leq t \wedge 75 \leq x^{pre} \wedge x^{pre} \leq 100 \wedge \\
& x^{pre} - 10t \leq x \wedge x \leq x^{pre} - 4t \wedge y = 50 + 10t \wedge \\
& 50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100) \\
\overset{elim.x^{pre}}{=}& (l_1, \boxed{\exists t}.\ 0 \leq t \wedge 75 \leq x + 10t \wedge x + 4t \leq 100 \wedge \\
& \boxed{y = 50 + 10t} \wedge \\
& 50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100) \\
\overset{elim.t}{=}& (l_1, \underbrace{125 \leq x + y \wedge x + 0.4y \leq 120 \wedge 50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100}_{=:\varphi^{R''}})
\end{aligned}
$$

$$
\begin{aligned}
\{R''\} \hat{\cap} P^{bad} \neq \emptyset \ \textit{iff}\ & \exists x.\exists y. \\
& \underbrace{125 \leq x + y \wedge x + 0.4y \leq 120 \wedge 50 \leq x \wedge x \leq 100 \wedge 50 \leq y \wedge y \leq 100}_{\varphi^{R''}} \wedge \\
& \underbrace{80 \leq x \wedge x \leq 100 \wedge 80 \leq y \wedge y \leq 100}_{\varphi^{bad}} \\
=& \boxed{\exists x}.\exists y.\ \underline{125 \leq x + y} \wedge \overline{x + 0.4y \leq 120} \wedge \\
& \underline{80 \leq x} \wedge \overline{x \leq 100} \wedge 80 \leq y \wedge y \leq 100 \\
\overset{elim.x}{=}& \exists y. 25 \leq y \wedge y \leq 100 \wedge 8\frac{1}{3} \leq y \wedge 80 \leq y \wedge y \leq 100 \\
=& \boxed{\exists y}.\underline{80 \leq y} \wedge \overline{y \leq 100} \\
\overset{elim.y}{=}& \textit{true} \rightarrow \{R^1_{l_1}\} \hat{\cap} P^{bad} \neq \emptyset
\end{aligned}
$$

Thus there exists a possibility of a rescue.

b) We can use the previously computed constraints to visualize the reachable state space: