Modeling and Analysis of Hybrid Systems

Convex polyhedra

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Informatik 2 - Theory of Hybrid Systems
RWTH Aachen University

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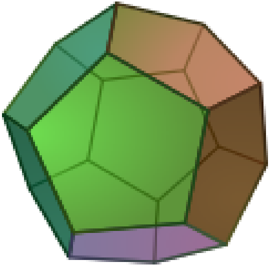
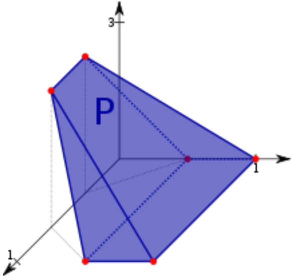
Contents

Polyhedra





Convex polyhedra

Definition

A polyhedron in $\mathbb{R}^d$ is the solution set to a finite number of linear inequalities with real coefficients in d real variables. A bounded polyhedron is called polytope.

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$$\forall x, y \in S.\ \forall \lambda \in [0, 1] \subseteq \mathbb{R}.\ \lambda x + (1 - \lambda) y \in S.$$

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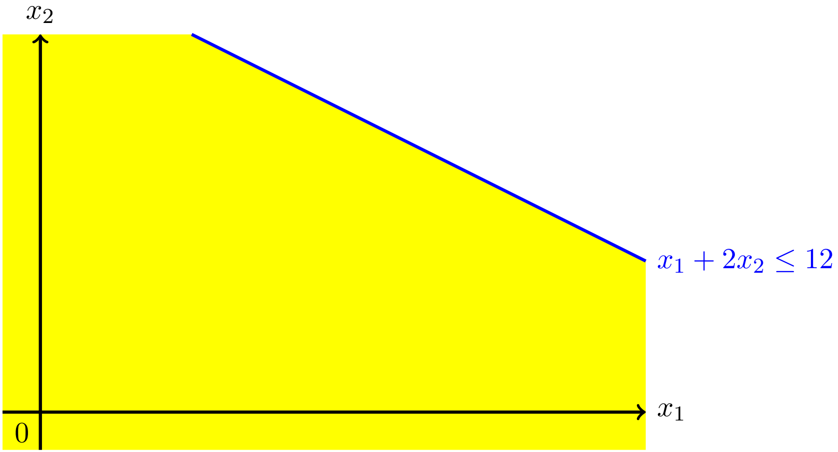
Depending on the form of the representation we distinguish between

- $\mathcal{H}$-polytopes and
- $\mathcal{V}$-polytopes

Intersection of a finite set of halfspaces



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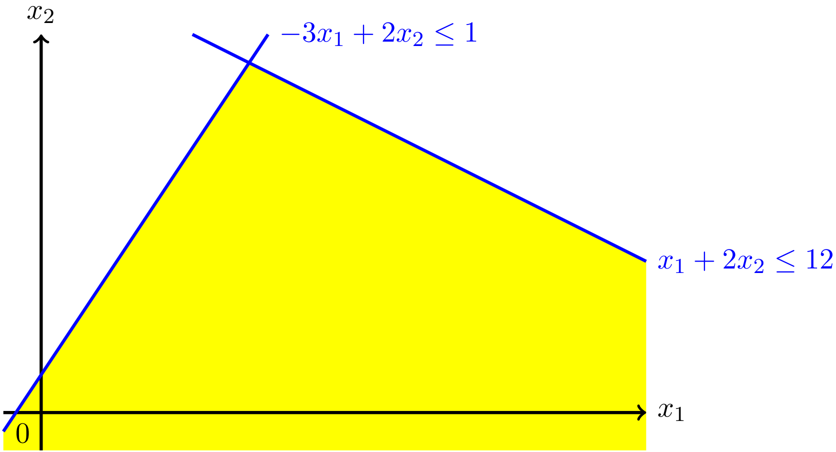

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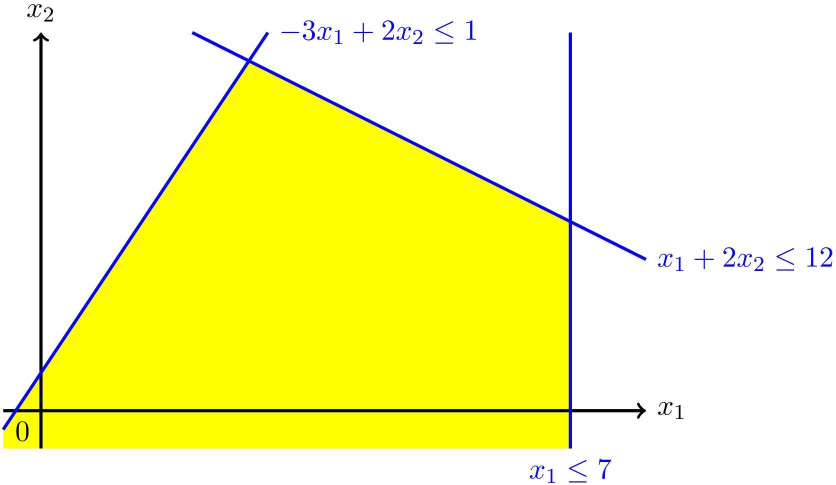

$\mathcal{H}$-polytopes

Definition (Closed halfspace)

A d-dimensional closed halfspace is a set $\mathcal{H} = \{x \in \mathbb{R}^d \mid c^T x \leq z\}$ for some $c \in \mathbb{R}^d$, called the normal of the halfspace, and a $z \in \mathbb{R}$.

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Definition ($\mathcal{H}$-polyhedron, $\mathcal{H}$-polytope)

A d-dimensional $\mathcal{H}$-polyhedron $P = \bigcap_{i=1}^{n} \mathcal{H}_i$ is the intersection of finitely many closed halfspaces. A bounded $\mathcal{H}$-polyhedron is called an $\mathcal{H}$-polytope.

The facets of a d-dimensional $\mathcal{H}$-polytope are $d-1$-dimensional $\mathcal{H}$-polytopes.

$\mathcal{H}$-polytopes

An $\mathcal{H}$-polytope

$$P \quad = \quad \bigcap_{i=1}^{n} \mathcal{H}_i \quad = \quad \bigcap_{i=1}^{n} \{x \in \mathbb{R}^d \mid c_i \cdot x \leq z_i\}$$

can also be written in the form

$$P = \{x \in \mathbb{R}^d \mid Cx \leq z\}.$$

We call (C, z) the $\mathcal{H}$-representation of the polytope.

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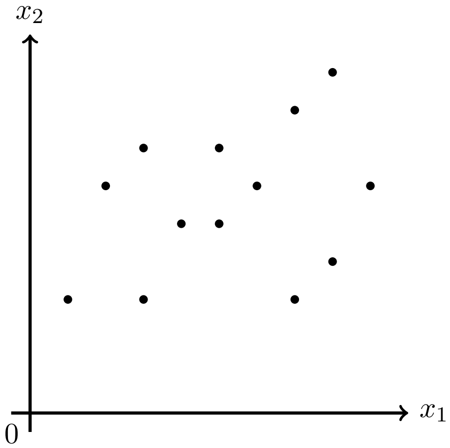
- An $\mathcal{H}$-polytope P has a finite number of vertices $V(P)$.

Convex hull of a finite set of points



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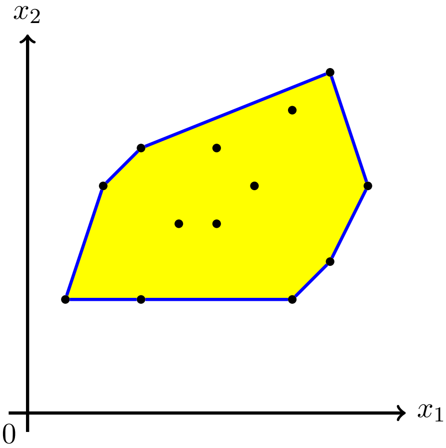

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$$conv(V) = \{x \in \mathbb{R}^d \mid \exists \lambda_1, \ldots, \lambda_n \in [0,1] \subseteq \mathbb{R}. \sum_{i=1}^{n} \lambda_i = 1 \wedge \sum_{i=1}^{n} \lambda_i v_i = x\}.$$

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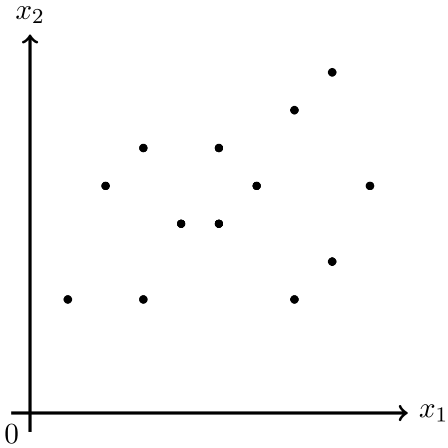
Unbounded polyhedra can be represented by extending convex hulls with conical hulls.

Conical hull of a finite set of points



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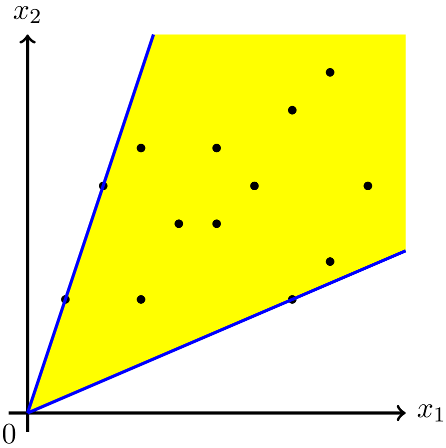

Conical hull of a finite set of points

If $U = \{u_1, \ldots, u_n\}$ is a finite set of points in $\mathbb{R}^d$, the conical hull of U is defined by

$$cone(U) = \{x \mid x = \sum_{i=1}^{n} \lambda_i u_i, \lambda_i \geq 0\}. \tag{1}$$

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Each polyhedra $P \subseteq \mathbb{R}^d$ can be represented by two finite sets $V, U \subseteq \mathbb{R}^d$ such that

$$P = conv(V) \oplus cone(U) \ .$$

If U is empty then P is bounded (e.g., a polytope).

Motzkin's theorem

- For each $\mathcal{H}$-polytope, the convex hull of its vertices defines the same set in the form of a $\mathcal{V}$-polytope, and vice versa,
- each set defined as a $\mathcal{V}$-polytope can be also given as an $\mathcal{H}$-polytope by computing the halfspaces defined by its facets.

The translations between the $\mathcal{H}$- and the $\mathcal{V}$-representations of polytopes can be exponential in the state space dimension d.

Contents

Operations

If we represent reachable sets of hybrid automata by polytopes, we need some operations like

- membership computation,
- intersection, or the
- union of two polytopes.

Operations: Membership

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  Alternatively: convert the $\mathcal{V}$-polytope into an $\mathcal{H}$-polytope by computing its facets.

Intersection

Intersection for two polytopes P_1 and P_2:

- $\mathcal{H}$-polytopes defined by $C_1x \le z_1$ and $C_2x \le z_2$:

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- $\mathcal{V}$-polytopes defined by V_1 and V_2: Convert P_1 and P_2 to $\mathcal{H}$-polytopes and convert the result back to a $\mathcal{V}$-polytope.

Union

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- $\mathcal{V}$-polytopes defined by V_1 and V_2:
  $\mathcal{V}$-representation $V_1 \cup V_2$.
- $\mathcal{H}$-polytopes defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$:

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- $\mathcal{V}$-polytopes defined by V_1 and V_2:
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- $\mathcal{H}$-polytopes defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$:
  convert to $\mathcal{V}$-polytopes and compute back the result.

Hardness of the convex hull computation

	$\mapsto$	$conv$	$\oplus$	$\cap$
$\mathcal{V}$-polytope	easy	–	–	–
$\mathcal{H}$-polytope	hard	–	–	–
$\mathcal{V}$-polytope and $\mathcal{V}$-polytope	–	easy	easy	hard
$\mathcal{H}$-polytope and $\mathcal{H}$-polytope	–	hard	hard	easy
$\mathcal{V}$-polytope and $\mathcal{H}$-polytope	–	hard	hard	hard

It could also be hard to translate a $\mathcal{V}$-polytope to an $\mathcal{H}$-polytope or vice versa.