Modeling and Analysis of Hybrid Systems

What's decidable about hybrid automata?

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Informatik 2 - Theory of Hybrid Systems
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Literature

Henzinger et al.: What's decidable about hybrid automata?

Journal of Computer and System Sciences, 57:94–124, 1998

Motivation

- The special class of timed automata with TCTL is decidable, thus model checking is possible.
- What about more expressive model classes for hybrid systems?

What is decidable about hybrid automata?

Two central problems for the analysis of hybrid automata:

- Safety: The problem to decide whether something "bad" can happend during the execution of a system.
- Liveness: The problem to decide whether there is always the possibility that something "good" will eventually happen during the execution of a system.

Both problems are decidable in certain special cases, and undecidable in certain general cases.

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- all conditions, effects, and flows are described by rectagular sets.

Definition

- A set $\mathcal{R} \subset \mathbb{R}^n$ is rectangular if it is a cartesian product of (possibly unbounded) intervals, all of whose finite endpoints are rationals.
- The set of rectangular sets in $\mathbb{R}^n$ is denoted $\mathcal{R}^n$.

Rectangular automaton

Definition

A rectangular automaton A is a tuple
$\mathcal{H} = (Loc, Var, Con, Lab, Edge, Act, Inv, Init)$ with

- finite set of locations Loc,
- finite set of real-valued variables $Var = \{x_1, \ldots, x_n\}$,
- a function Con $: Loc \to 2^{Var}$ assigning controlled variables to locations,
- finite set of synchronization labels Lab,
- finite set of edges $Edge \subseteq Loc \times Lab \times \mathcal{R}^n \times \mathcal{R}^n \times 2^{\{1,\ldots,n\}} \times Loc$,
- a flow function $Act : Loc \to \mathcal{R}^n$,
- an invariant function $Inv : Loc \to \mathcal{R}^n$,
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- Is the state space rectangular?

Rectangular automaton

- Flows: first time derivatives of the flow trajectories in location $l \in Loc$ are within $Act(l)$
- Jumps: $e = (l, a, pre, post, jump, l') \in Edge$ may move control from location l to location l' starting from a valuation in pre, changing the value of each variable x_i to a nondeterministically chosen value from $post_i$ (the projection of $post$ to the ith dimension), such that the values of the variables $x_i \notin jump$ are unchanged.

Operational semantics

Operational semantics

$$\frac{\begin{array}{c}(l, a, \textit{pre}, \textit{post}, \textit{jump}, l') \in Edge \\ \vec{x} \in \textit{pre} \quad \vec{x}' \in \textit{post} \quad \forall i \notin \textit{jump}.\ x_i' = x_i \quad \vec{x}' \in Inv(l')\end{array}}{(l, \vec{x}) \stackrel{a}{\rightarrow} (l', \vec{x}')} \quad \texttt{Rule}_{\texttt{Discrete}}$$

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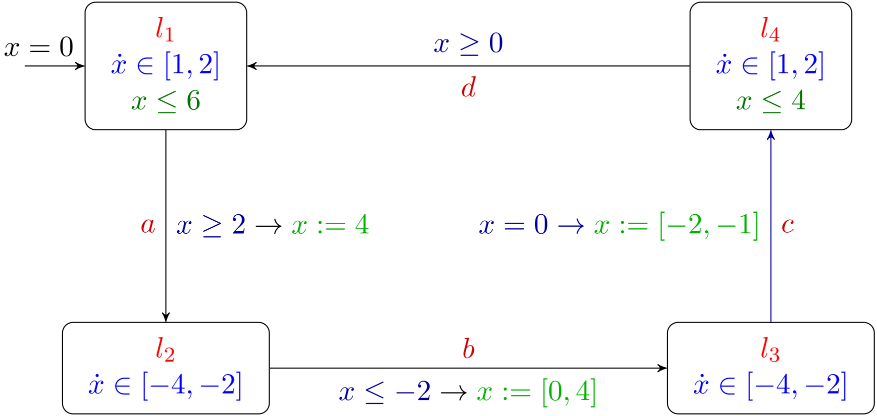
- Execution step: $\rightarrow = \xrightarrow{a} \cup \xrightarrow{t}$
- Path: $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \vec{x}_0)$, $\vec{x}_0 \in Inv(l_0)$
- Initial path: path $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \vec{x}_0)$, $\vec{x}_0 \in Init(l_0)$
- Reachability of a state: exists an initial path leading to the state

Example rectangular automaton



Remarks

- If we replace rectangular sets with linear sets, we obtain linear hybrid automata, a super-class of rectangular automata.
- A timed automaton is a special rectangular automaton.

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- A timed automaton is a special rectangular automaton.

This class lies at the boundary of decidability.

Decidability

The reachability problem is decidable for initialized rectangular automata:

Decidability

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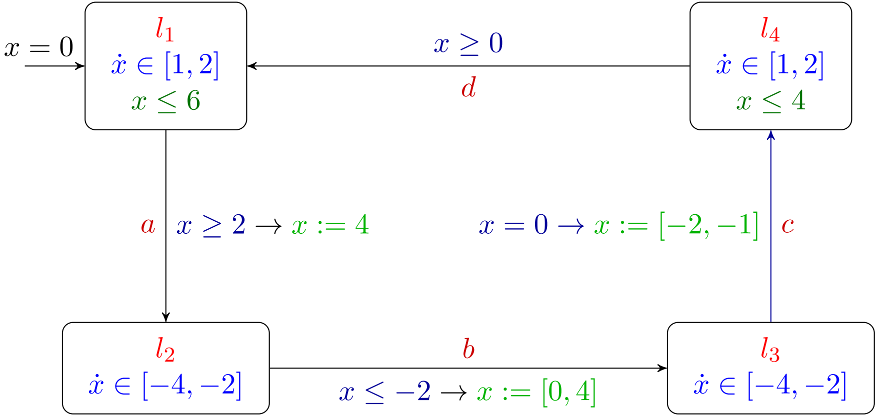
Definition

A rectangular automaton A is initialized, if for every edge $(l, a, pre, post, jump, l')$ of A, and every variable index $i \in \{1, \ldots, n\}$ with $Act(l)_i \neq Act(l')_i$, we have that $i \in jump$.

The reachability problem becomes undecidable if one of the restrictions is relaxed.

Initialized rectangular automaton



This rectangular automaton is initialized.

What we already know

A timed automaton is a special rectangular automaton such that

- for each edge, *post*$_i$ is a single value for each $i \in$ *jump* and
- every variable is a clock, i.e., $Act(l)(x) = [1, 1]$ for all locations l and variables x.

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Lemma

The reachability problem for timed automata is complete for PSPACE.

Decidability results

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$$
\begin{array}{c}
\text{Timed automaton} \\
\uparrow \\
\text{Initialized stopwatch automaton} \\
\uparrow \\
\text{Initialized singular automaton} \\
\uparrow \\
\text{Initialized rectangular automaton}
\end{array}
$$

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

Initialized stopwatch automata

- A stopwatch is a variable with derivatives 0 or 1 only.
- A stopwatch automaton is as a timed automaton but allowing stopwatch variables instead of clocks.
- Initialized stopwatch automata can be polynomially encoded by timed automata.

Lemma

The reachability problem for initialized stopwatch automata is complete for PSPACE.

However, the reachability problem for non-initialized stopwatch automata is undecidable.

Proof idea:

Proof idea: Notice, that a timed automaton is a stopwatch automaton such that every variable is a clock.
Assume that C is an n-dimensional initialized stopwatch automaton. Let κ_C be the set of constants used in the definition of C, and let $\kappa_- = \kappa_C \cup \{-\}$.
We define an n-dimensional timed automaton D_C with locations $Loc_{D_C} = Loc_c \times \kappa_-^{1,\ldots,n}$. Each location (l, f) of D_C consists of a location l of C and a function $f : \{1, \ldots, n\} \to \kappa_-$. Each state $q = ((l, f), \vec{x})$ of D_C represents the state $\alpha(q) = (l, \vec{y})$ of C, where $y_i = x_i$ if $f(i) = -$, and $y_i = f(i)$ if $f(i) \neq -$.
Intuitively, if the ith stopwatch of C is running (slope 1), then its value is tracked by the value of the ith clock of D_C; if the ith stopwatch is halted (slope 0) at value $k \in \kappa_C$, then this value is remembered by the current location of D_C.

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

$\uparrow$

Initialized singular automaton

Initialized singular automata

- A variable x_i is a finite-slope variable if $\mathit{flow}(l)_i$ is a singleton in all locations l.
- A singular automaton is as a stopwatch automaton but allowing finite-slope variables instead of stopwatches.
- Initialized singular automata can be polynomially encoded by initialized stopwatch automata.

Lemma

The reachability problem for initialized singular automata is complete for PSPACE.

Proof idea:

Proof idea: Let B be an n-dimensional initialized singular automaton. We define an n-dimensional initialized stopwatch automaton C_B with the same location set, edge set, and label set as B.
Each state $q = (l, \vec{x})$ of C_B corresponds to the state $\beta(q) = (l, \beta(\vec{x}))$ of B with $\beta : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows:
For each location l of B, if $Act_B(l) = \Pi_{i=1}^{n}[k_i, k_i]$, then $\beta(x_1, \ldots, x_n) = (l_1 \cdot x_1, \ldots, l_n \cdot x_n)$ with $l_i = k_i$ if $k_i \neq 0$, and $l_i = 1$ if $k_i = 0$;
β can be viewed as a rescaling of the state space. All conditions in the automaton B occur accordingly rescaled in C_B.
We have:

- The reachable set of $Reach(B)$ of B is $\beta(Reach(C_B))$.

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

$\uparrow$

Initialized singular automaton

$\uparrow$

Initialized rectangular automaton

Lemma

The reachability problem for initialized rectangular automata is complete for PSPACE.

Proof idea:

Proof idea: An n-dimensional initialized rectangular automaton A can be translated into a $2n$-dimensional initialized singular automaton B, such that B contains all reachability information about A.
The translation is similar to the subset construction for determinizing finite automata.
The idea is to replace each variable c of A by two finite-slope variables c_l and c_u: the variable c_l tracks the least possible value of c, and c_u tracks the greatest possible value of c.