# Modeling and Analysis of Hybrid Systems

Lecture Notes

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Theory of Hybrid Systems
Informatik 2

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# Contents

# Basic Notations

We use the following standard notations:

| **Symbol** | **Description** | |
|---|---|---|
| *Sets and set operators* | | |
| $A \cup B$ | $= \{x \mid x \in A \vee x \in B\}$ | set union |
| $A \cap B$ | $= \{x \mid x \in A \wedge x \in B\}$ | set intersection |
| $A \backslash B$ | $= \{x \mid x \in A \wedge x \notin B\}$ | set minus |
| $A \times B$ | $= \{(a, b) \mid a \in A \wedge b \in B\}$ | cross product of two sets $A$ and $B$ |
| $\mathbb{Z}$ | | set of integers |
| $\mathbb{N}$ | | set of natural numbers including 0 |
| $\mathbb{N}^d$ | $= \underbrace{\mathbb{N} \times \ldots \times \mathbb{N}}_{d\ times}$ | the $n$-dimensional space of natural numbers |
| $\mathbb{N}_{>0}$ | $= \mathbb{N} \backslash \{0\}$ | set of positive natural numbers |
| $\mathbb{Q}$ | | set of rational numbers |
| $\mathbb{R}$ | | set of real numbers |
| $\mathbb{R}^d$ | $= \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{d\ times}$ | the $d$-dimensional real space |
| $\mathbb{R}_{\geq 0}$ | $= \{x \in \mathbb{R} \mid x \geq 0\}$ | set of non-negative real numbers |
| $\mathbb{R}_{>0}$ | $= \{x \in \mathbb{R} \mid x > 0\}$ | set of positive real numbers |
| $2^M$ | $= \{P \mid P \subseteq M\}$ | powerset of the set $M$ |
| *Mappings* | | |
| $id$ | $: M \rightarrow M$ | identity mapping for a set $M$ with $id(m) = m$ for all $m \in M$ |
| $f(M)$ | $= \{f(m) \mid m \in M\} \subseteq D_2$ | image of a set $M \subseteq D_1$ according to a mapping $f : D_1 \rightarrow D_2$ |

# Chapter 1

# Introduction

Most areas of computer science deal with *discrete systems*, i.e., systems whose evolution can be described by a sequence of discrete state changes. Prominent examples are programs executing a sequence of computation steps, each of them possibly modifying the program's heap and stack. When we are only interested in the program's input-output behavior, such computation steps can be seen as instantaneous, *discrete* state changes.

Whereas the modeling and the analysis of discrete systems is a typical computer science subject, the development of methods and tools for the modeling and simulation of *dynamic systems* is hosted mainly in physics and control theory. Quantities of dynamic systems, like the temperature of a room or the speed of an object, evolve *continuously* over time according to the laws of physics in dependence on the current system state and the influence of the environment.

Discrete systems (e.g., sensors, chips, computer programs) are often used to control the behavior (e.g., temperature, speed, acceleration) of dynamic systems. The resulting system, consisting of the controller together with the controlled system, exhibits a *combined discrete and continuous behavior*, and is therefore called a *hybrid system*. Hybrid systems can have quite complex behavior, posing a challenging task for their analysis.

This book is devoted to modeling formalisms and algorithmic analysis techniques for different classes of hybrid systems, from the view point of computer science. It can be used as learning material for undergraduate courses, but also as a state-of-the-art overview for graduate students and researchers.

The contents of the book are as follows:

- We start with introducing *hybrid systems* on a number of examples in Chapter 2.
- In Chapter 3 we recall some automata-based *modeling* approaches for discrete systems, in the absence of dynamic behavior. We use *labeled state transition systems* (*Kripke structures*) to model finite-state systems, and *labeled transition systems* to model general discrete systems with possibly infinite state spaces. We define the *temporal logics* LTL, CTL, and CTL$^*$ to specify properties of discrete systems, and give a short introduction to (explicit) CTL *model checking* for labeled state transition systems.
- We discuss *discrete-time systems* in a nutshell in Chapter 4 before we deal with continuous-time systems in the following chapters.
- *Timed automata* [AD94, BK08], extending discrete systems with a notion of time, are introduced in Chapter 5. We use the timed temporal logic TCTL to specify properties of timed automata. We show that the validity of TCTL properties for timed automata is decidable by giving the standard model checking algorithm.

- Timed automata are quite restrictive in their modeling power. In Chapter 6 we define *rectangular automata*, a bit more general class, which is at the boundary of decidability: though checking TCTL properties of *initial* rectangular automata is a decidable problem, relaxing any of the restrictions on the expressivity of the modeling language leads to undecidability. We give the decidability proof following [HKPV98] in form of a reduction to TCTL model checking for timed automata.
- In Chapter **??** we elaborate on the above-mentioned border of decidability and give some *undecidability results* via selected reduction proofs from [HKPV98] and [ACH$^+$95] for different model classes.
- Even if the reachability problem for more expressive modeling formalisms for hybrid systems is in general undecidable, we need them to model more complex systems without too strong abstraction. Though undecidability implies that we cannot give any complete model checking algorithm for them, there might exist useful incomplete algorithms for their analysis. Such a more expressive model class is given by *linear hybrid automata*[1], being the subject of Chapter 7. They are particularly interesting, because the bounded reachability problem (reachability within a fixed finite number of steps) is still decidable and efficiently computable for this class. We discuss a fixed-point-based algorithm from [ACH$^+$95] and mention some approximation and abstraction techniques.
- To model hybrid systems more precisely, in Chapter 8 we introduce *hybrid automata.* The dynamics in these models is specified by ordinary differential equations, which might be linear or non-linear. The reachability analysis for hybrid automata requires special (over-approximative) representation techniques for state sets. We discuss representations by different geometric objects like convex polyhedra, oriented rectangular hulls, zonotopes, support functions and orthogonal polyhedra. Using such representations, we discuss an incomplete fixed-point-based algorithm for the reachability analysis of hybrid automata.

Regarding undergraduate courses, the contents are determined such that they demonstrate the application and usefulness of a wide range of general computer science methods and techniques:

- *Formal modeling:* labeled state transition systems (Section 3.1.1), labeled transition systems (Section 3.1.2), discrete-time models (Chapter 4), timed automata (Section 5.1), rectangular automata (Section 6.1), linear hybrid automata (Section 7.1), general hybrid automata (Section 8.1);
- *Logics to formalize system properties:* propositional logic (Section 3.2.1), temporal logics LTL, CTL, CTL$^*$ (Section 3.2.2), timed temporal logic TCTL (Section 5.2);
- *Decidability issues:* proving decidability constructively by giving a finite bisimulation-based abstraction (Section 5.3 for timed automata), proving decidability by reducing the question to a known problem (Section 6.2 for initialized rectangular automata), proving undecidability by reduction to a known problem (Chapter **??** for 2-rate singular automata);
- *Model checking:* CTL properties for labeled state transition systems (Section 3.3), TCTL properties for timed automata (Section 5.3), fixedpoint-based reachability analysis (Sections 7.2, 7.3 and 8.2), minimization (Section 7.5);
- *Approximation:* for state sets of linear (Section 7.4) and general (Section 8.2) hybrid automata.

[1] There are two different notions of linear hybrid automata. We mean here systems with a linear behavior, and not with linear differential equations describing the continuous behavior.

# Chapter 2

# Hybrid Systems

*Discrete systems* are systems with discrete, *instantaneous* state changes. E.g., when abstracting away physical details, a sensor reporting whether a tank is full or whether the temperature is above a certain threshold can be considered as a simple discrete system. Also a program running on a computer can be seen as a discrete system, when we assume that each atomic execution step changes the program's configuration in a discrete manner. Note that, though the state space of a program can be very large, due to the finite memory it is finite. Other systems might have an infinite or even uncountable state space, they are nevertheless classified as discrete systems when their *state changes* can be assumed to be discrete.

*Dynamic systems* are systems with a real-valued state space and *continuous* behavior. Physical systems with quantities like time, temperature, speed, acceleration etc. are dynamic systems. There evolution over time can be described by continuous functions or ordinary differential equations.

*Hybrid systems* are systems with combined discrete and continuous behavior (cf. Figure 2.1). Typical examples are physical systems controlled by a discrete controller. In modern cars there are hundreds of embedded digital chips helping to drive the car, that means, controlling the physical behavior like speed and acceleration. Behind the autopilot of an airplane there is a program running on a computer and acting with the physical environment.

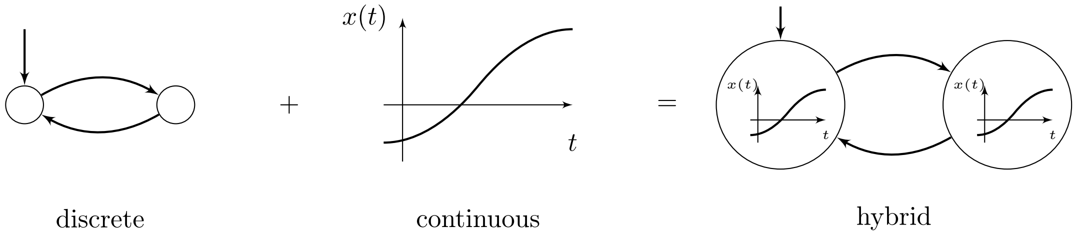


Figure 2.1: Hybrid systems exhibit a combined discrete-continuous behavior

In the following we introduce some hybrid system examples from [ACH$^+$95, Hen96].

**Example 2.1 (Thermostat).** *Assume a thermostat, which senses the temperature $x$ of a room and turns a heater on and off in order to keep the temperature between $17°C$ and $23°C$. Initially, the heater is on and the temperature is $20°C$. If the heater is on, the temperature increases according to the differential equation $\dot{x} = K(h - x)$ where $h \in \mathbb{R}_{>0}$ is a constant of the heater* *and $K \in \mathbb{R}_{>0}$ is a room constant. If the temperature is $22°C$ or above, but at latest when it reaches $23°C$, the heater gets turned off. If the heater is off, the temperature falls according to the differential equation $\dot{x} = -Kx$. If the temperature falls to $18°C$ or below, but at latest if it reaches $17°C$, the heater gets switched on. Figure 2.2 visualizes a possible behavior of the system, both of its continuous dynamics (the temperature) and its discrete control (the heater being on or off).*

*This system is hybrid. The discrete part of the system's state consists of the control mode of the heater being on or off. The continuous part is the temperature which continuously evolves over time, taking values from $\mathbb{R}$. The discrete part controls the continuous part by changing the discrete state and thereby influencing the continuous behavior.*

*Note that, since the heater gets switched on and off within certain temperature intervals, the system is non-deterministic. Replacing these intervals by fixed values would yield a deterministic system.*

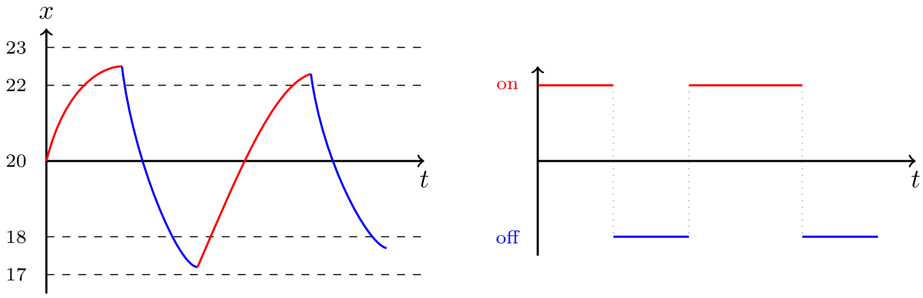


Figure 2.2: A possible behavior of the thermostat: the continuous dynamics for the temperature (left) and the control state with the heater being on or off (right) as a function of time

**Example 2.2 (Water-level monitor).** *Assume two identical, constantly leaking water tanks and a hose that refills exactly one of the tanks at each point in time (Figure 2.3 left). Let us denote the water level in the two tanks by $x_1$ and $x_2$, respectively, and let the leaking lead to a decrease of $v_1$ and $v_2$ units of tank height per time unit, respectively, for some $v_1, v_2 \in \mathbb{R}_{>0}$ without refilling. The hose fills $w \in \mathbb{R}_{>0}$ units of tank height per time unit. Thus the derivative of the water height for the first tank is $\dot{x}_1 = w - v_1$ when it gets refilled and $\dot{x}_1 = -v_1$ otherwise. The water height in the second tank changes according to $\dot{x}_2 = w - v_2$ when it gets refilled and $\dot{x}_2 = -v_2$ otherwise. When refilling the first tank, the hose switches to the second tank when its water level $x_2$ reaches a given lower threshold $r_2 \in \mathbb{R}_{>0}$. The switch from the second tank to the first one works analogously when $x_1$ reaches some $r_1 \in \mathbb{R}_{>0}$.*

*Also this is a hybrid system. The discrete part of the state space consists of the position of the hose refilling either the first or the second tank. The continuous part of the state space corresponds to the water heights in the tanks which evolve continuously over time.*

**Example 2.3 (Bouncing ball).** *Assume a bouncing ball with the initial height $h \in \mathbb{R}_{\geq 0}$ and with an initial upwards directed speed $v \in \mathbb{R}_{>0}$. Due to gravity, the ball has the acceleration $\dot{v} = -g$. Thus the ball's speed is decreasing to $0$ until the ball reaches its highest position, and gets negative when the ball is falling down again. The ball bounces when it reaches the earth at position $h = 0$ with a speed $v < 0$. When bouncing, the sign of $v$ gets inverted, and a part of the ball's kinetic energy gets lost. Its speed after bouncing is $-cv$ with some $c \in (0, 1) \subseteq \mathbb{R}$ and $v$ the*

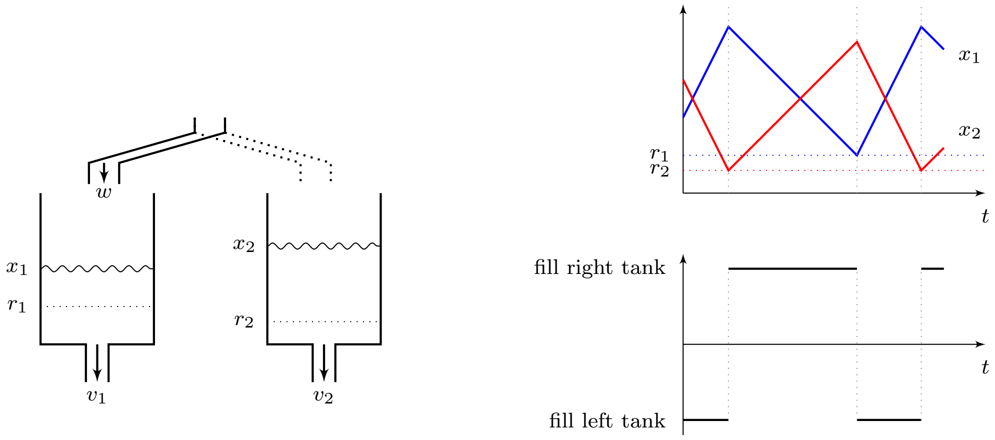


Figure 2.3: Water-level monitor illustration (left) and a possible behavior (right)

*speed before bouncing. Figure 2.4 illustrates the behavior of the system.*

*The continuous part of the state space covers the physical quantities of height and speed which follow the same evolution rules all the time. Thus there is only a single mode ("moving") for the ball behavior, and the state space does not have any discrete component. However, the discrete time points of bouncing introduce discrete events. That's why a bouncing ball can also be considered as a hybrid system.*

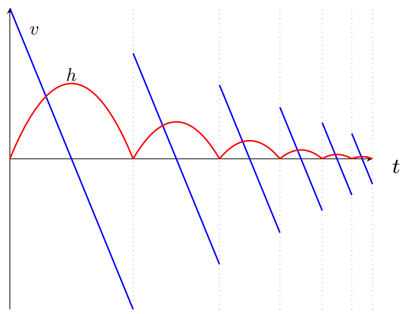


Figure 2.4: A possible behavior of the bouncing ball

## Exercises

**Exercise 2.1.** Give three further examples for hybrid systems.

**Exercise 2.2.** Assume a vending machine, where you can buy some beverage after having inserted some coins. Which parts of the machine's state components can be considered as discrete and which are continuous?

**Exercise 2.3.** *Real-time systems* are systems whose correct functioning requires (under others) that they react within certain time limits. Since time is a continuous quantity, real-time systems are also hybrid systems. Give some examples for real-time systems!

**Exercise 2.4.** Assume an ice block, which gets warmed by the sun from $-10^\circ$ to $10^\circ$. What is hybrid in this process? Explain the discrete and the continuous components of the state of the ice.

# Chapter 3

# Discrete Models

In the next chapters we address the modeling and analysis of hybrid systems. Before doing so, in this chapter we first recall some fundamentals about the *modeling* of discrete systems in Section 3.1 and about *logics* that allow to formalize properties of discrete systems in Section 3.2. Finally, we explain the basic idea of (explicit) *CTL model checking* for discrete finite-state systems in Section 3.3.

## 3.1 Modeling Languages

As *modeling languages* we use in this chapter *labeled state transition systems* (generally known as *Kripke structures*, see Section 3.1.1) and *labeled transition systems* (Section 3.1.2) which additionally allow variables in the model.

### 3.1.1 Labeled State Transition Systems (LSTSs)

Labeled state transition systems consist of a set of states, a set of initial states where the execution starts, and labeled transitions between the states.

**Definition 3.1 (Syntax of labeled state transition systems).** *A* labeled state transition system (LSTS) *is a tuple* $\mathcal{LSTS} = (\Sigma, Lab, Edge, Init)$ *with* *LSTS* $\mathcal{LSTS}$

- *a (possibly infinite) set* $\Sigma$ *of* states*,* $\Sigma$
- *a set* $Lab$ *of (*synchronization*)* labels*,* *Lab*
- *a set* $Edge \subseteq \Sigma \times Lab \times \Sigma$ *of labeled* transitions *or* edges*, and* *Edge*
- *a non-empty set* $Init \subseteq \Sigma$ *of* initial states*.* *Init*

The semantics allows to build paths of an LSTS starting in an initial state and following transitions.

**Definition 3.2 (Semantics of LSTS).** *The* operational semantics *of a labeled state transition system* $\mathcal{LSTS} = (\Sigma, Lab, Edge, Init)$ *is given by the following single rule:*

$$\frac{(\sigma, a, \sigma') \in Edge}{\sigma \overset{a}{\to} \sigma'}\ \mathit{Rule}_{discrete}$$

*We call* $\sigma \overset{a}{\to} \sigma'$ *an* (execution) step*. A* path *(or* run *or* execution*)* $\pi$ *of* $\mathcal{LSTS}$ *is a (finite or infinite) sequence* $\pi = \sigma_0 \overset{a_0}{\to} \sigma_1 \overset{a_1}{\to} \ldots$. $\pi$

*For a path $\pi = \sigma_0 \to \sigma_1 \to \ldots$ of $\mathcal{LSTS}$ and some $i \in \mathbb{N}$, $i \leq |\pi|$, let $\pi(i) = \sigma_i$ and $\pi^i = \sigma_i \to \sigma_{i+1} \to \ldots$.*

$\Pi(\sigma)$ *We use $\Pi_{\mathcal{LSTS}}$ (or simply $\Pi$) to denote the set of all paths of $\mathcal{LSTS}$ and define $\Pi_{\mathcal{LSTS}}(\sigma) = \{\pi \in \Pi_{\mathcal{LSTS}} \mid \pi(0) = \sigma\}$.*
*The path $\pi$ is* initial *if $\pi(0)$ is an initial state. A state is* reachable *iff there is an initial path leading to it.*

We sometimes simply write $\pi = \sigma_0 \to \sigma_1 \to \ldots$ when the labels of the edges are not of interest. We say that $\sigma' \in \Sigma$ is a *successor* of $\sigma \in \Sigma$ and $\sigma$ is a *predecessor* of $\sigma'$ iff $\sigma \overset{a}{\to} \sigma'$ for some $a \in Lab$. Note that for a path $\pi = \sigma_0 \overset{a_0}{\to} \sigma_1 \overset{a_1}{\to} \ldots$ the state $\sigma_{i+1}$ is a successor of $\sigma_i$ for each $0 \leq i < |\pi|$, where $|\pi|$ denotes the number of steps in the path (possibly being infinity).

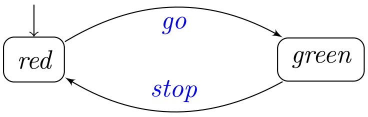
The labels of the set *Lab* are attached to edges and are used for synchronization purposes in the parallel composition (see page 14).

To be able to formalize properties of LSTSs, it is common to define a set of *atomic propositions* $AP$ and a labeling function $L : \Sigma \to 2^{AP}$ assigning a set of atomic propositions to each state. The set $L(\sigma) \subseteq AP$ consists of all propositions that are defined to hold in the state $\sigma$. These propositional labels on states should not be mixed up with the synchronization labels on edges.

$AP, L$

A labeled state transition system $\mathcal{LSTS} = (\Sigma, Lab, Edge, Init)$ can be represented as a directed graph, where the vertices of the graph are the states from $\Sigma$ and the (labeled) edges are the transitions from *Edge*. The initial states are marked by an incoming edge without source.

**Example 3.1 (Pedestrian light).** *We model a pedestrian traffic light in a crossing by a labeled state transition system $\mathcal{LSTS} = (\Sigma, Lab, Edge, Init)$. The light can be red or green (we do not model the light being off or blinking). Thus we can represent the light's state set by $\Sigma = \{red, green\}$. Assume the light is initially red, i.e., $Init = \{red\}$. Possible state changes go from red to green and from green to red, yielding $Edge = \{(red, go, green), (green, stop, red)\}$ for a possible label set $Lab = \{go, stop\}$. The labels can be used, e.g., to synchronize state changes with another light in the same crossing. The model $\mathcal{LSTS}$ can be visualized as follows:*



*This model is* deterministic*, i.e., both of its states has a single possible successor state. The system has the single initial run $red \overset{go}{\to} green \overset{stop}{\to} red \ldots$.*

Larger or more complex systems are often modeled componentwise, such that the whole system is given by the *parallel composition* of the components. Component-local, non-synchronizing transitions, having labels belonging to one component's label set only, are executed in an interleaved manner. Synchronizing transitions of the components, agreeing on the label, are executed synchronously.

**Definition 3.3 (Parallel composition of LSTS).** *Let*

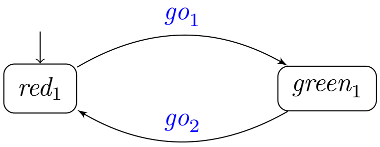
$$\begin{array}{rcl} \mathcal{LSTS}_1 & = & (\Sigma_1, Lab_1, Edge_1, Init_1) \text{ and} \\ \mathcal{LSTS}_2 & = & (\Sigma_2, Lab_2, Edge_2, Init_2) \end{array}$$

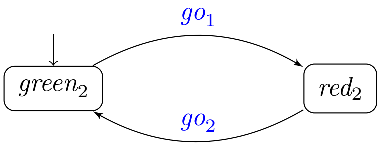
$\mathcal{LSTS}_1 || \mathcal{LSTS}_2$ *be two LSTSs. The* parallel composition *$\mathcal{LSTS}_1 || \mathcal{LSTS}_2 = (\Sigma, Lab, Edge, Init)$ is an LSTS with*

- $\Sigma = \Sigma_1 \times \Sigma_2$,
- $Lab = Lab_1 \cup Lab_2$,
- $((s_1, s_2), a, (s'_1, s'_2)) \in Edge$ *iff*
  1. $a \in Lab_1 \cap Lab_2$, $(s_1, a, s'_1) \in Edge_1$, *and* $(s_2, a, s'_2) \in Edge_2$, *or*
  2. $a \in Lab_1 \backslash Lab_2$, $(s_1, a, s'_1) \in Edge_1$, *and* $s_2 = s'_2$, *or*
  3. $a \in Lab_2 \backslash Lab_1$, $(s_2, a, s'_2) \in Edge_2$, *and* $s_1 = s'_1$,
- $Init = Init_1 \times Init_2$.

To demonstrate the advantages of compositional modeling, we give an example for the parallel composition of two traffic lights.

**Example 3.2 (Two pedestrian lights).** *Assume now a crossing of two roads with two pedestrian lights, similar to those from Example 3.1, one in north-south and one in east-west direction. The two lights are composed such that they allow pedestrians to pass alternatingly.*





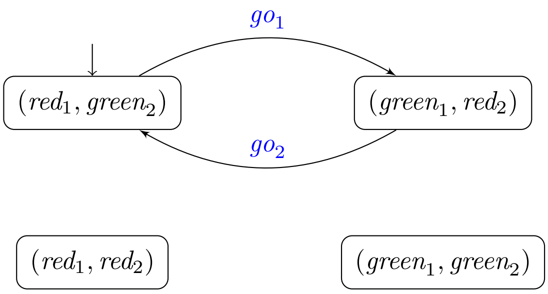
*Formally, the two LSTSs are given by*

$$\begin{aligned}
\mathcal{LSTS}_1 &= (\underbrace{\{red_1, green_1\}}_{\Sigma_1}, \underbrace{\{go_1, go_2\}}_{Lab_1}, \underbrace{\{(red_1, go_1, green_1), (green_1, go_2, red_1)\}}_{Edge_1}, \underbrace{\{red_1\}}_{Init_1}) \\
\mathcal{LSTS}_2 &= (\underbrace{\{red_2, green_2\}}_{\Sigma_2}, \underbrace{\{go_1, go_2\}}_{Lab_2}, \underbrace{\{(red_2, go_2, green_2), (green_2, go_1, red_2)\}}_{Edge_2}, \underbrace{\{green_2\}}_{Init_2})
\end{aligned}$$

*The parallel composition* $\mathcal{LSTS}_1 || \mathcal{LSTS}_2 = (\Sigma, Lab, Edge, Init)$ *is by definition:*

$$\begin{aligned}
\Sigma &= \{(green_1, green_2), (green_1, red_2), (red_1, green_2), (red_1, red_2)\} \\
Lab &= \{go_1, go_2\} \\
Edge &= \{((red_1, green_2), go_1, (green_1, red_2)), ((green_1, red_2), go_2, (red_1, green_2))\} \\
Init &= \{(red_1, green_2)\}
\end{aligned}$$

*The parallel composition can be visualized be the following graph:*



*Note that the states* $(red_1, red_2)$ *and* $(green_1, green_2)$ *are not reachable, i.e., the two lights are never green respectively red at the same time.*

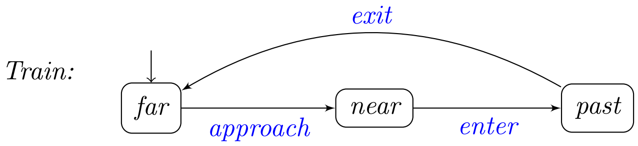
*Again, the composition is deterministic and has a single initial run* $(red_1, green_2) \overset{go_1}{\to} (green_1, red_2) \overset{go_2}{\to} (red_1, green2) \ldots$.

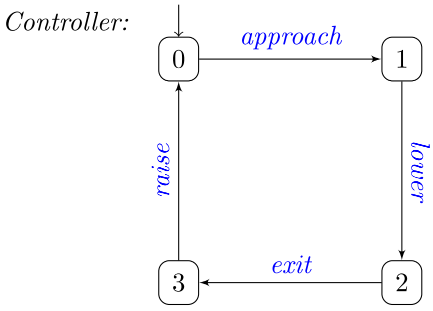
Another well-known example for the parallel composition is that of a railway crossing.

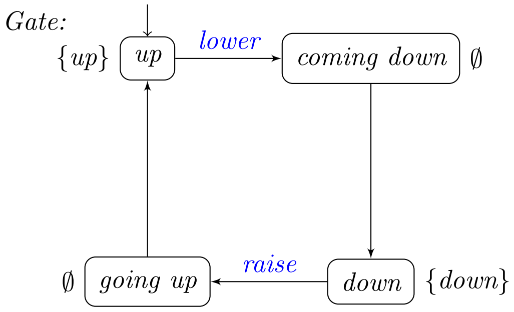
**Example 3.3 (Railroad crossing).** *Assume the crossing of a railroad with a street, secured by a gate. The system consists of three components: a train, a controller and a gate. The train communicates with the controller, and the controller communicates with the gate as follows.*

- *Sensors recognize when the train is approaching to the gate and an "approach" signal is sent to the controller. Similarly, if the train has left the railroad crossing, an "exit" signal gets sent to the controller.*
- *The controller reacts to an incoming "approach" signal with the sending of a "lower" signal to the gate. Similarly, when the controller receives an "exit" signal, it sends a "raise" signal to the gate.*
- *The gate reacts to an incoming "lower" signal with closing the gate, and to a "raise" signal with opening the gate.*

*If we are interested in the communication aspects only, we can model the railroad crossing system as the parallel composition of the following three LSTS components:*







*Given the proposition set* $AP = \{up, down\}$*, we can define a state labeling function* $L$ *assigning a set of propositions to the states of the gate as depicted in the above graph.*

*The formal specification of the parallel composition is the content of Exercise 3.3. The parallel composition's initial state is* $(far, 0, up)$*. In the initial state the gate cannot execute, because the only possible transition from the state up has the label lower that synchronizes with the controller, but the controller first has to move to the state 1 to be able to synchronize on it. Thus first synchronization on approach must take place. Therefore, each initial path of the composition starts with the step* $(far, 0, up) \overset{approach}{\to} (near, 1, up)$.

### 3.1.2 Labeled Transition Systems (LTSs)

Another, more expressive but still discrete modeling language are *labeled transition systems* (*LTSs*), which additionally allow *variables* in the model. Here we consider real-valued variables only, and in the following we restrict the formalisms accordingly.

Given a set of real-valued variables $Var$, a *valuation* is a function $\nu : Var \to \mathbb{R}$ assigning values to the variables. We use $V_{Var}$ (or short $V$) to denote the set of all valuations for the variable set $Var$. $Var, \nu, V$

An LTS has a finite set of *locations*, also called *modes*, which can possibly be entered with different valuations. The current *state* $\sigma = (\ell, \nu)$ of an LTS is determined by the current location $l$ and the current valuation $\nu$. A set of *initial states* specifies the states in which the execution may start. $Loc$ $\sigma, \Sigma$

The locations of an LTS are connected by labeled *transitions* (edges). In contrast to LSTSs, each edge of an LTS can have a guard and an effect, specified in form of a *transition relation* $\mu \subseteq V \times V$: the transition can be taken with a valuation $\nu$ thereby changing the valuation to $\nu'$ iff $(\nu, \nu') \in \mu$.

**Example 3.4.** *Assume a variable set* $Var = \{x\}$ *and a transition that is enabled if* $x > 0$ *holds and it decreases the value of* $x$ *by* $1$. *The corresponding transition relation would be* $\mu = \{(\nu, \nu') \in V^2 \mid \nu(x) > 0 \land \nu'(x) = \nu(x) - 1\}$.

In the following definition of LTSs we also embed *controlled variables* and *$\tau$-transitions* (also called *stutter* transitions). Their role will become clear later when we define the parallel composition of LTSs. Intuitively, these constructs help us to define "local", "output" or "write" variables of an LTS whose values may not be changed by non-synchronizing steps of other parallel LTSs.

**Definition 3.4 (Syntax of labeled transition systems).** *A* labeled transition system (LTS) *is a tuple* $\mathcal{LTS} = (Loc, Var, Con, Lab, Edge, Init)$ *with* LTS, $\mathcal{LTS}$

- *a finite set Loc of* locations*,*
- *a finite set Var of real-valued* variables*,*
- *a function* $Con : Loc \to 2^{Var}$ *assigning a set of* controlled variables *to each location,* $Con$
- *a finite set Lab of labels, including the* stutter label $\tau \in Lab$, $\tau \in Lab$
- *a finite set* $Edge \subseteq Loc \times Lab \times 2^{V^2} \times Loc$ *of* edges *or* transitions *including a $\tau$-transition* $(\ell, \tau, Id, \ell)$ *for each location* $\ell \in Loc$ *with* $Id = \{(\nu, \nu') \in V^2_{Var} \mid \forall x \in Con(\ell).\ \nu'(x) = \nu(x)\}$, *and where all edges with label $\tau$ are $\tau$-transitions, and* $Edge$ $Id$
- *a set* $Init \subseteq \Sigma$ *of* initial states*,*

*where* $\Sigma = Loc \times V_{Var}$ *denotes the state space of* $\mathcal{LTS}$.

**Definition 3.5 (Semantics of LTSs).** *The* operational semantics *of a labeled transition system* $\mathcal{LTS} = (Loc, Var, Con, Lab, Edge, Init)$ *is specified by the following single rule:*

$$\frac{(\ell, a, \mu, \ell') \in Edge \quad (\nu, \nu') \in \mu}{(\ell, \nu) \xrightarrow{a} (\ell', \nu')}\ \mathit{Rule}_{discrete}$$

*We call* $\sigma \xrightarrow{a} \sigma'$ *an* (execution) step*, which we also write as* $\sigma \to \sigma'$ *when we are not interested in its label. A* path *(or* run *or* execution*)* $\pi$ *of* $\mathcal{LTS}$ *is a (finite or infinite) sequence* $\pi = \sigma_0 \xrightarrow{a_0} \sigma_1 \xrightarrow{a_1} \ldots$*; for* $i \in \mathbb{N}$, $i \leq |\pi|$, *we define* $\pi(i) = \sigma_i$ *and* $\pi^i = \sigma_i \to \sigma_{i+1} \to \ldots$. $\pi, \pi(i), \pi^i$
*We use* $\Pi_{\mathcal{LTS}}$ *(or simply* $\Pi$*) to denote the set of all paths of* $\mathcal{LTS}$ *and define* $\Pi_{\mathcal{LTS}}(\sigma) = \{\pi \in \Pi_{\mathcal{LTS}} \mid \pi(0) = \sigma\}$. $\Pi(\sigma), \Pi$

```
      method mult(int y, int z){
         int x;
ℓ0       x := 0;
ℓ1       while( y > 0 ) {
ℓ2          y := y-1;
ℓ3          x := x+z;
         }
ℓ4    }
```

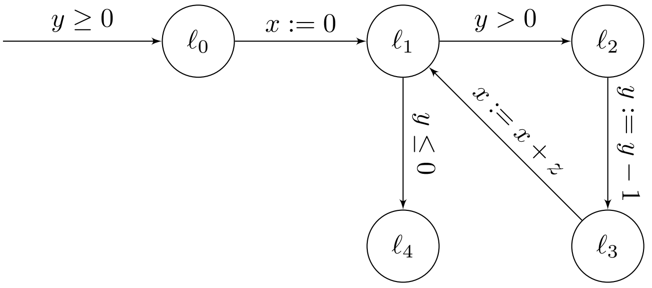


Figure 3.1: Modeling a simple while program with an LTS.

*The path $\pi$ is* initial *if $\pi(0)$ is an initial state. A state is* reachable *iff there is an initial path leading to it.*

We sometimes simply write $\pi = \sigma_0 \to \sigma_1 \to \ldots$ when the labels of the edges are not of interest. Note again that for a path $\pi = \sigma_0 \overset{a_0}{\to} \sigma_1 \overset{a_1}{\to} \ldots$ there is a transition $\sigma_i \overset{a_i}{\to} \sigma_{i+1}$ between all successive states $\sigma_i$ and $\sigma_{i+1}$, $0 \leq i < |\pi|$, in the path, where $|\pi|$ denotes the number of steps in the path (possibly being infinity).

Based on the operational semantics, an LTS induces an underlying LSTS state space model: a transition $(\ell, \nu) \overset{a}{\to} (\ell', \nu')$ can be performed in the induced LSTS if there is an edge $(\ell, a, \mu, \ell')$ from $\ell$ to $\ell'$ in the LTS with $(\nu, \nu') \in \mu$.

The next example shows how LTSs can be used to describe program execution.

**Example 3.5 (Modeling a simple while program).** *The simple while program on the left of Figure 3.1 calculates $x := y \cdot z$ for two input integers $y$ and $z$ with $y \geq 0$. Each instruction corresponds to a transition $(\ell, a, \mu, \ell')$ with the source location $l$ being the program location before the instruction, the target location $l'$ being the program location after the instruction, a label $a$ which is omitted here because no synchronization is needed, and a set of valuation pairs $\mu$ describing the condition or effect represented by the instruction.*

*Formally, this (closed) system can be defined as a transition system $\mathcal{LTS} = (Loc, Var, Con, Lab, Edge, Init)$ where*

- $Loc = \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4\}$,
- $Var = \{x, y, z\}$,
- $V = \{\nu \mid \nu : Var \to \mathbb{R}\}$ *and* $\Sigma = Loc \times V$,
- $Con(\ell) = Var$ *for each* $\ell \in Loc$,
- $Lab = \{a, \tau\}$,
- $Edge =$
$$\begin{array}{lllr} \{(\ell_0, & a, & \{(\nu,\nu') \in V^2 \mid \nu'(x) = 0 \wedge \nu'(y) = \nu(y) \wedge \nu'(z) = \nu(z)\}, & \ell_1), \\ \ (\ell_1, & a, & \{(\nu,\nu') \in V^2 \mid \nu(y) > 0 \wedge \nu' = \nu\}, & \ell_2), \\ \ (\ell_2, & a, & \{(\nu,\nu') \in V^2 \mid \nu'(x) = \nu(x) \wedge \nu'(y) = \nu(y) - 1 \wedge \nu'(z) = \nu(z)\}, & \ell_3), \\ \ (\ell_3, & a, & \{(\nu,\nu') \in V^2 \mid \nu'(x) = \nu(x) + \nu(z) \wedge \nu'(y) = \nu(y) \wedge \nu'(z) = \nu(z)\}, & \ell_1), \\ \ (\ell_1, & a, & \{(\nu,\nu') \in V^2 \mid \nu(y) \leq 0 \wedge \nu' = \nu\}, & \ell_4), \\ \ \tau_{\ell_0}, \tau_{\ell_1}, \tau_{\ell_2}, \tau_{\ell_3}, \tau_{\ell_4}\}, & & & \end{array}$$
- $Init = \{(\ell_0, \nu) \in \Sigma \ \mid \ \nu(y) \in \mathbb{N} \wedge \nu(z) \in \mathbb{Z}\}$

*with $\tau_{\ell_i} = (\ell_i, \tau, \{(\nu, \nu') \in V^2 \mid \nu = \nu'\}, \ell_i)$ for all $l \in Loc$. This LTS model is illustrated on the right of Figure 3.1 (without showing the transition label $a$ and the $\tau$-transitions).*

The *parallel composition* of LTSs allows to model larger systems compositionally. Intuitively, two LTSs running in parallel may execute non-synchronizing steps interleaved on their own, whereas synchronizing steps are executed simultaneously in both components. Whether a step is synchronizing or not depends on the fact whether both systems have the step's label in their label sets. One of the components can take a transition with a common label only if the other component also takes a transition with the same label. For this joint step the conditions and effects of both transitions must be considered, i.e., the transition relation for the joint step is the intersection of the transition relations of both local transitions.

If one of the components executes a local, non-synchronizing step, the other component is basically not active. However, in the parallel composition of LTSs we define the other component to take a so-called *$\tau$-transition* or *stutter transition*, a "do nothing" step. The reason for this is twofold: Firstly, this makes the definitions and the underlying algorithms more unique, since in each step both systems take a transition. Secondly, and more importantly, sometimes we would like to define components with variables local to this component, or with variables that can only be read but not written by the other components. Then the $\tau$-transitions of this component will specify in their transition relation that the values of those variables are not modified by the environment's non-synchronizing steps. Variables that a component has under its control and that must not be modified by the local steps of its environment are defined by the function *Con*.

**Definition 3.6 (Parallel composition of LTSs).** *Let*

$$\begin{array}{rcl}\mathcal{LTS}_1 & = & (Loc_1, Var, Con_1, Lab_1, Edge_1, Init_1) \text{ and}\\ \mathcal{LTS}_2 & = & (Loc_2, Var, Con_2, Lab_2, Edge_2, Init_2)\end{array}$$

*be two LTSs.The* parallel composition *or* product

$$\mathcal{LTS} = \mathcal{LTS}_1 \mid\mid \mathcal{LTS}_2 = (Loc, Var, Con, Lab, Edge, Init)$$

$\mathcal{LTS}_1 \mid\mid \mathcal{LTS}_2$

*of $\mathcal{LTS}_1$ and $\mathcal{LTS}_2$ is the LTS defined by*

- $Loc = Loc_1 \times Loc_2$,
- $Con((\ell_1, \ell_2)) = Con_1(\ell_1) \cup Con_2(\ell_2)$,
- $Lab = Lab_1 \cup Lab_2$,
- $((\ell_1, \ell_2), a, \mu, (\ell_1', \ell_2')) \in Edge$ *iff*
  - *there exist* $(\ell_1, a_1, \mu_1, \ell_1') \in Edge_1$ *and* $(\ell_2, a_2, \mu_2, \ell_2') \in Edge_2$ *such that*
  - *either* $a_1 = a_2 = a$ *or*
    $a_1 = a \in Lab_1 \backslash Lab_2$ *and* $a_2 = \tau$, *or*
    $a_1 = \tau$ *and* $a_2 = a \in Lab_2 \backslash Lab_1$, *and*
  - $\mu = \mu_1 \cap \mu_2$, *and*
- $Init = \{((\ell_1, \ell_2), \nu) \mid (\ell_1, \nu) \in Init_1 \wedge (\ell_2, \nu) \in Init_2\}$.

**Example 3.6.** *Assume the parallel composition of the following two LTSs:*

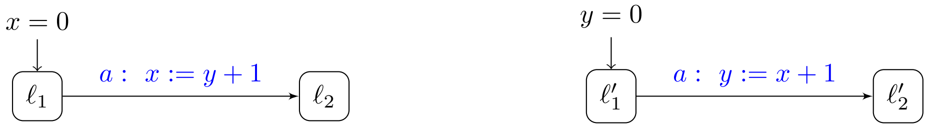
$$\begin{array}{rcl}\mathcal{LTS}_1 & = & (Loc_1, Var, Con_1, Lab_1, Edge_1, Init_1)\\ \mathcal{LTS}_2 & = & (Loc_2, Var, Con_2, Lab_2, Edge_2, Init_2)\end{array}$$

*with*

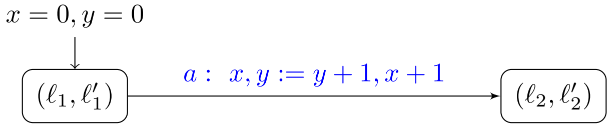
- $Loc_1 = \{\ell_1, \ell_2\}$, $Loc_2 = \{\ell_1', \ell_2'\}$,
- $Var = \{x, y\}$,
- $Con_1(\ell_1) = Con_1(\ell_2) = \{x\}$, $Con_2(\ell_1') = Con_2(\ell_2') = \{y\}$,

- $Lab_1 = Lab_2 = \{a, \tau\}$,
- $Edge_1 = \{(\ell_1,\ a,\ \{(\nu,\nu') \in V^2 \mid \nu'(x) = \nu(y)+1\},\ \ell_2),\ \tau_{\ell_1},\ \tau_{\ell_2}\}$,
  $Edge_2 = \{(\ell'_1,\ a,\ \{(\nu,\nu') \in V^2 \mid \nu'(y) = \nu(x)+1\},\ \ell'_2),\ \tau_{\ell'_1},\ \tau_{\ell'_2}\}$,
- $Init_1 = \{(\ell_1, \{\nu \in V | \nu(x) = 0\})\}$, $Init_2 = \{(\ell'_1, \{\nu \in V | \nu(y) = 0\})\}$

*with* $\tau_\ell = \{(\nu,\nu') \in V^2 \mid \forall v \in Con_i(\ell).\ \nu(v) = \nu'(v)\}$ *for all* $i = 1, 2$ *and* $\ell \in Loc_i$. *Graphically (without representing the control variables and* $\tau$*-transitions):*

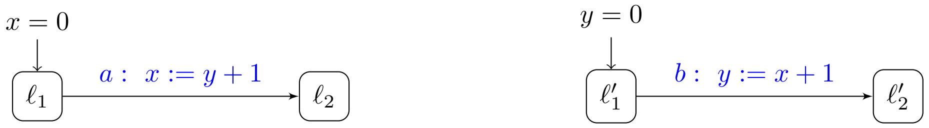


*The graphical representation of the parallel composition (again without control variables and* $\tau$*-transitions) looks as follows:*

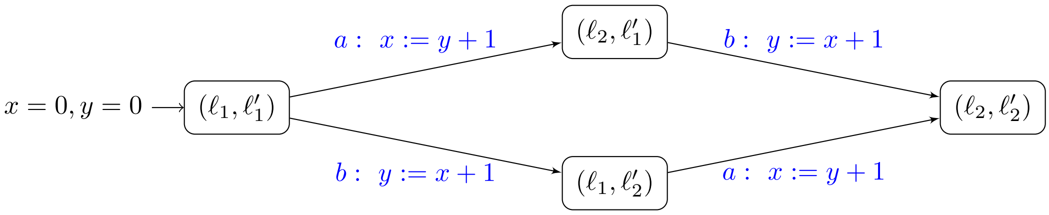


*As the only non-*$\tau$*-transitions of the LTSs are synchronized by the label* $a$*, all runs of the system are of the form* $\sigma_0 \xrightarrow{\tau} \ldots \sigma_0 \xrightarrow{a} \sigma_1 \xrightarrow{\tau} \sigma_1 \ldots$ *with* $\sigma_0(x) = \sigma_0(y) = 0$ *and* $\sigma_1(x) = \sigma_1(y) = 1$.

*Let us modify the example such that the transitions do not synchronize:*



*The parallel composition looks as follows:*



*Now the transitions interleave, and if we skip the transitions where both components do a* $\tau$*-step, we get two possible runs:*

- $\sigma_0 \xrightarrow{a} \sigma_1 \xrightarrow{b} \sigma_2$ *with*
  - $\sigma_0 = ((\ell_1, \ell'_1), \nu_0)$, $\nu_0(x) = \nu_0(y) = 0$,
  - $\sigma_1 = ((\ell_2, \ell'_1), \nu_1)$, $\nu_1(x) = 1$, $\nu_1(y) = 0$, *and*
  - $\sigma_2 = ((\ell_2, \ell'_2), \nu_2)$, $\nu_2(x) = 1$, $\nu_2(y) = 2$, *or*
- $\sigma_0 \xrightarrow{b} \sigma_1 \xrightarrow{a} \sigma_2$ *with*
  - $\sigma_0 = ((\ell_1, \ell'_1), \nu_0)$, $\nu_0(x) = \nu_0(y) = 0$,
  - $\sigma_1 = ((\ell_1, \ell'_2), \nu_1)$, $\nu_1(x) = 0$, $\nu_1(y) = 1$, *and*
  - $\sigma_2 = ((\ell_2, \ell'_2), \nu_2)$, $\nu_2(x) = 2$, $\nu_2(y) = 1$.

## 3.2 Logics

For the formalization of properties for discrete systems, after introducing *propositional logic* in a nutshell in Section 3.2.1 we deal with the temporal logics LTL, CTL and CTL$^*$ in Section 3.2.2.

### 3.2.1 Propositional Logic

Assume a set of states $\Sigma$, a set of atomic propositions $AP$, and a labeling function $L : \Sigma \to 2^{AP}$ assigning to each state a set of propositions holding in that state. Then we can use *propositional logic* to describe properties of states. Propositional logic formulae are built from atomic propositions using Boolean operators according to the following abstract syntax:

$$\varphi \ ::= \ a \mid (\varphi \wedge \varphi) \mid (\neg\varphi)$$

$\wedge, \neg$

with $a \in AP$ and where $\wedge$ is the "and"-operator for conjunction and $\neg$ is the operator for negation. As syntactic sugar the constants *true* and *false*, and further Boolean operators like $\vee$ ("or"), $\to$ ("implies"), $\leftrightarrow$ ("if and only if"), etc. can be introduced. We often omit parentheses with the convention that the strength of binding is in the order $\neg, \wedge, \vee, \to, \leftrightarrow$, i.e., $\neg$ binds the strongest and $\leftrightarrow$ the weakest. We use $Form_{prop}^{AP}$ (or short $Form_{prop}$) to denote the set of all propositional logic formulae over the atomic proposition set $AP$.

$\vee, \to, \leftrightarrow$

$Form_{prop}$

Propositional logic formulae are evaluated in the context of a state with the help of the labeling function. The semantics is given by the relation $\models_{prop} \ \subseteq \ \Sigma \times Form_{prop}$ (or short $\models$), which is defined recursively over the structure of propositional logic formulae as follows:

$\models_{prop}$

$$\begin{array}{lll}
\sigma \models_{prop} a & \textit{iff} & a \in L(\sigma), \\
\sigma \models_{prop} (\varphi_1 \wedge \varphi_2) & \textit{iff} & \sigma \models_{prop} \varphi_1 \ \textit{and} \ \sigma \models_{prop} \varphi_2, \\
\sigma \models_{prop} (\neg\varphi) & \textit{iff} & \sigma \not\models_{prop} \varphi.
\end{array}$$

Though propositional logic is well-suited to describe *states* of a system, we are also interested in describing *computations* of systems. Propositional logic extended with temporal modalities can be used for this purpose.

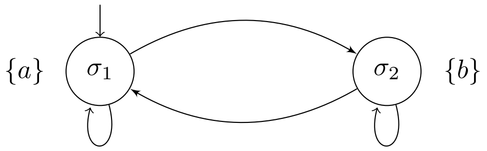
### 3.2.2 Temporal Logics

Assume in the following a labeled state transition system $\mathcal{LSTS} = (\Sigma, \mathit{Lab}, \mathit{Edge}, \mathit{Init})$, a set of atomic propositions $AP$, and a labeling function $L : \Sigma \to 2^{AP}$. The semantics of $\mathcal{LSTS}$ specifies its behavior as a set of paths. This path set can also be seen as a set of trees, which we get by sharing common prefixes of paths and branching only at the place of the first difference. I.e., for each initial state there is a *computation tree* and each path of the system corresponds to a path in one of the trees.

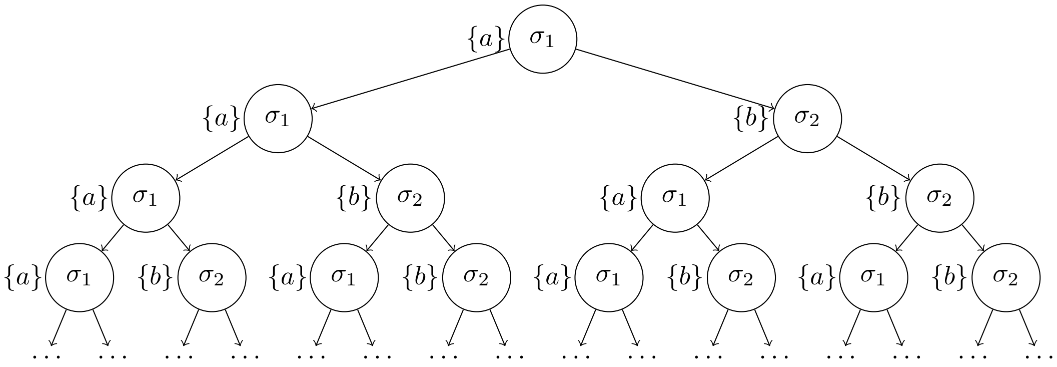
For deterministic systems with a single initial state, the computation tree is just a line. For non-deterministic systems, the branching at a node of a computation tree represents possible non-deterministic choices for further execution. Each reachable state is represented as a node in one of the trees as many times as the number of different (finite) paths leading to it. Note that this might happen infinitely often when the state is part of a reachable loop.

In the following we assume deadlock-free systems, i.e., infinite computation trees.

**Example 3.7 (Computation tree).** *Assume the following simple state transition system, where we omit synchronization labels on edges, but depict the labeling of states with atomic propositions:*



*This system has the following computation tree:*



Next we describe the temporal logics LTL, CTL, and CTL$^*$, which are suited to argue about paths in the computation tree.

### Linear Temporal Logic (LTL)

*Linear Temporal Logic (LTL)* is suited to argue about single (linear) paths in the computation tree.

**Definition 3.7 (Syntax of LTL).** *Assume a set AP of atomic propositions. LTL has the abstract* syntax

$\mathcal{X}$,$\mathcal{U}$

$$\varphi \quad ::= \quad a \mid (\varphi \wedge \varphi) \mid (\neg\varphi) \mid (\mathcal{X}\varphi) \mid (\varphi\, \mathcal{U}\, \varphi)$$

$Form_{LTL}$

*where $a \in AP$. We use $Form_{LTL}^{AP}$ (or short $Form_{LTL}$) to denote the set of LTL formulae over $AP$.*

Again, we omit parentheses when it causes no confusion, assuming that the Boolean operators bind stronger than the temporal ones.

Recall that for a path $\pi = \sigma_0 \to \sigma_1 \to \ldots$ of $\mathcal{LSTS}$ and some $i \in \mathbb{N}$, $i \leq |\pi|$, we defined $\pi(i) = \sigma_i$ and $\pi^i = \sigma_i \to \sigma_{i+1} \to \ldots$ (see Definition 3.2).

A path $\pi$ satisfies a proposition $a \in AP$ if the proposition holds in the first state $\pi(0)$ of $\pi$, i.e., if $a \in L(\pi(0))$. Using the "next time" temporal operator $\mathcal{X}$ we can build LTL formulae $\mathcal{X}\varphi$ ("next time $\varphi$") which are satisfied by a path $\pi$ iff $\varphi$ holds in $\pi^1$, i.e., when removing the first state from $\pi$. The second temporal operator is the "until" operator. The formula $\varphi_1 \mathcal{U}\ \varphi_2$ ("$\varphi_1$ until $\varphi_2$") is satisfied by a path $\pi = \sigma_0 \to \sigma_1 \to \ldots$ iff $\varphi_2$ holds for some suffix $\pi^j$ and $\varphi_1$ holds all the time before, i.e., for all $\pi^i$ with $0 \leq i < j$.

$\mathcal{F}$, $\mathcal{G}$

As syntactic sugar the temporal operators $\mathcal{F}$ ("finally" or "eventually") and $\mathcal{G}$ ("globally") can be introduced. The formula $\mathcal{F}\varphi$ ("finally $\varphi$) is defined as $true\mathcal{U}\,\varphi$, stating that $\varphi$ will be true after a finite number of steps. The formula $\mathcal{G}\varphi$ ("globally $\varphi$") is defined as $\neg(true\, \mathcal{U}\, \neg\varphi)$, stating that $\varphi$ holds all along the path.

**Remark 3.1.** Some approaches define two further temporal operators which we do not use in the following but mention them for completeness. The first one is the "release" operator: $\varphi_1 \mathcal{R} \varphi_2$, defined as $\neg((\neg\varphi_1)\mathcal{U}(\neg\varphi_2))$, expresses that $\varphi_2$ holds either forever or until $\varphi_1 \wedge \varphi_2$ gets valid. The second one is the "weak until": the formula $\varphi_1 \mathcal{U}_{weak} \varphi_2$, defined as $(\varphi_1 \mathcal{U} \varphi_2) \vee (\mathcal{G}\varphi_1)$, weakens the meaning of "until" with the possibility that $\varphi_1$ holds forever without $\varphi_2$ becoming true.

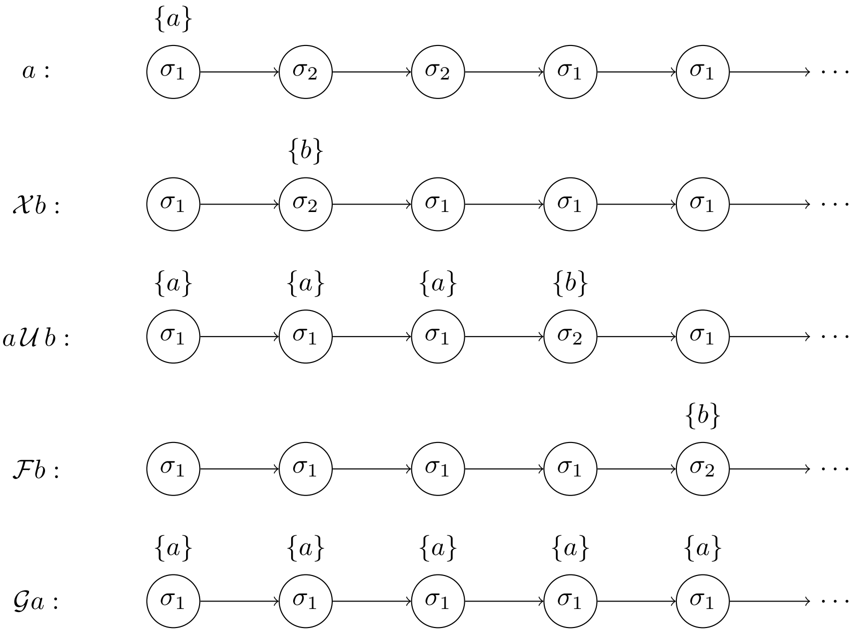
It is also possible to define operators "since", "previous", "once", "always been" referring to the past. They are symmetric to "until", "next", "finally" and "globally", but they refer to the past computation instead of the future one.

**Definition 3.8 (Semantics of LTL).** *Assume an atomic proposition set $AP$, a labeled state transition system $\mathcal{LSTS} = (\Sigma, Lab, Edge, Init)$ and a state labeling function $L : \Sigma \to 2^{AP}$. The* semantics *of LTL is given by the satisfaction relation $\models_{LTL} \subseteq (\Sigma \cup \Pi) \times Form_{LTL}$ (or short $\models$) which evaluates LTL formulae in the context of a path as follows:* $\models_{LTL}$

$$\begin{array}{lll}
\pi \models_{LTL} a & \textit{iff} & a \in L(\pi(0)) \\
\pi \models_{LTL} \varphi_1 \wedge \varphi_2 & \textit{iff} & (\pi \models_{LTL} \varphi_1) \wedge (\pi \models_{LTL} \varphi_2) \\
\pi \models_{LTL} \neg\varphi & \textit{iff} & \pi \not\models_{LTL} \varphi \\
\pi \models_{LTL} \mathcal{X}\varphi & \textit{iff} & \pi^1 \models_{LTL} \varphi \\
\pi \models_{LTL} \varphi_1 \mathcal{U} \varphi_2 & \textit{iff} & \exists j \geq 0.\ (\pi^j \models_{LTL} \varphi_2) \wedge \forall 0 \leq i < j.\ (\pi^i \models_{LTL} \varphi_1)\ .
\end{array}$$

*For a state $\sigma \in \Sigma$ and an LTL formula $\varphi$ we define $\sigma \models_{LTL} \varphi$ to hold iff $\pi \models_{LTL} \varphi$ for all paths $\pi \in \Pi(\sigma)$ of $\mathcal{LSTS}$ starting in $\sigma$, and $\mathcal{LSTS} \models_{LTL} \varphi$ iff $\sigma_0 \models_{LTL} \varphi$ for all $\sigma_0 \in Init$.*

**Example 3.8.** *We give some example LTL formulae and some paths of the system from Example 3.7 satisfying them. Thereby we omit labelings irrelevant for the satisfaction.*



*The initial state $\sigma_1$ of the system $\mathcal{LSTS}$ from Example 3.7 does not satisfy $\mathcal{F}b$ (written $\sigma_1 \not\models_{LTL} \mathcal{F}b$), since there is a path $\pi = \sigma_1 \to \sigma_1 \to \ldots$ on which $b$ never holds. But it satisfies $\mathcal{F}a$, since the proposition $a$ holds in the initial state.*

**Remark 3.2.** There are two special subclasses of path properties: safety and liveness properties. Intuitively, a *safety* property states that something “bad” never happens. E.g., the safety property $\mathcal{G}a$ expresses that the proposition $a$ holds all the time. Partial correctness of a program—whenever the program terminates its output is correct—is also a safety property. The violation of a safety property by a path $\pi$ can be shown by looking at a finite prefix of $\pi$.

In contrast, *liveness* properties express that something “good” will eventually happen. E.g., $\mathcal{F}a$ is a liveness property meaning that $a$ will happen after a finite number of steps. Termination is a typical liveness property. To show the violation of a liveness property we must consider infinite paths.

Note that not all path formulae starting with the “globally” operator are safety properties. E.g., reactiveness, stating that something (let’s say $a$) holds over and over again, is a typical liveness property which can be formalized as $\mathcal{GF}a$.

By definition, the sets of safety and liveness properties are disjoint. However, their union does not cover all path properties. E.g., program correctness can be expressed only by the conjunction of a safety property (partial correctness) and a liveness property (termination).

**Remark 3.3.** Besides the above notation, for the temporal operators there is another commonly used alternative notation:

| | | | |
|---|---|---|---|
| | $\mathcal{U}$ | for | $\mathcal{U}$ |
| $\bigcirc$ | $\bigcirc$ | for | $\mathcal{X}$ |
| $\Diamond$ | $\Diamond$ | for | $\mathcal{F}$ |
| $\Box$ | $\Box$ | for | $\mathcal{G}$ |

For example, the formula $\mathcal{GF}\varphi$ can also be written as $\Box\Diamond\varphi$.

### Computation Tree Logic (CTL)

Whereas LTL argues about linear paths, CTL formulae specify properties of computation trees. We distinguish between state formulae and path formulae. Intuitively, state formulae describe properties of the states (nodes) in the computation tree, and path formulae describe properties of paths in the tree. On the one hand, a path formula can be converted into a state formula by putting an existential or a universal quantifier in front of it, denoting that the path formula holds for a path respectively for all paths starting in a given node of the computation tree. On the other hand, state formulae are used to generate path formulae using the temporal operators. This implies, that a CTL state formula contains quantifiers and temporal operators in an alternating manner.

**Definition 3.9 (Syntax of CTL).** *Assume a set $AP$ of atomic propositions.* CTL state formulae *can be built according to the abstract grammar*

$$\psi \quad ::= \quad a \mid (\psi \wedge \psi) \mid (\neg\psi) \mid (E\varphi) \mid (A\varphi)$$

*with $a \in AP$ and where $\varphi$ is a CTL path formula.*
CTL path formulae *are built according to the abstract grammar*

$$\varphi \quad ::= \quad \mathcal{X}\psi \mid \psi\,\mathcal{U}\,\psi$$

*where $\psi$ is a CTL state formula.*
$Form_{CTL}$ CTL formulae *are CTL state formulae building the set $Form_{CTL}^{AP}$ (or short $Form_{CTL}$).*

We omit parentheses when it causes no confusion. Similarly to LTL, we can introduce the “finally” and “globally” operators. For state formulae $\psi$ we define $\mathcal{F}\psi = true\,\mathcal{U}\,\psi$ as path formulae.

Note that the LTL definition $\mathcal{G}\psi = \neg(true\,\mathcal{U}\,\neg\psi)$ of "globally" cannot be directly adapted to CTL, since it is not accepted by the CTL syntax. Instead, we define

$$\begin{array}{lcl} E\mathcal{G}\psi & \leftrightarrow & \neg A\,true\,\mathcal{U}\,\neg\psi \\ A\mathcal{G}\psi & \leftrightarrow & \neg E\,true\,\mathcal{U}\,\neg\psi\ . \end{array}$$

**Definition 3.10 (Semantics of CTL).** *Assume an atomic proposition set $AP$, a labeled state transition system $\mathcal{LSTS} = (\Sigma, Lab, Edge, Init)$ and a state labeling function $L : \Sigma \to 2^{AP}$. The satisfaction relation $\models_{CTL} \subseteq (\Sigma \cup \Pi) \times Form_{CTL}$ (or short $\models$) evaluates CTL state formulae in the context of a state, and CTL path formulae in the context of a path as follows:* $\models_{CTL}$

$$\begin{array}{lll} \sigma \models_{CTL} a & \textit{iff} & a \in L(\sigma) \\ \sigma \models_{CTL} \psi_1 \wedge \psi_2 & \textit{iff} & (\sigma \models_{CTL} \psi_1) \wedge (\sigma \models_{CTL} \psi_2) \\ \sigma \models_{CTL} \neg\psi & \textit{iff} & \sigma \not\models_{CTL} \psi \\ \sigma \models_{CTL} E\varphi & \textit{iff} & \exists \pi \in \Pi(\sigma).\ \pi \models_{CTL} \varphi \\ \sigma \models_{CTL} A\varphi & \textit{iff} & \forall \pi \in \Pi(\sigma).\ \pi \models_{CTL} \varphi \\ & & \\ \pi \models_{CTL} \mathcal{X}\psi & \textit{iff} & \pi(1) \models_{CTL} \psi \\ \pi \models_{CTL} \psi_1 \mathcal{U}\,\psi_2 & \textit{iff} & \exists 0 \le j.\ (\pi(j) \models_{CTL} \psi_2) \wedge \forall 0 \le i < j.\ (\pi(i) \models_{CTL} \psi_1)\ . \end{array}$$

*For $\psi \in Form_{CTL}$ we define $\mathcal{LSTS} \models_{CTL} \psi$ iff $\sigma_0 \models_{CTL} \psi$ for all $\sigma_0 \in Init$.*

**Example 3.9.** *For our $\mathcal{LSTS}$ from Example 3.7 the CTL formula $A\mathcal{G}E\mathcal{X}b$ holds, since at each node of the computation tree we can take a transition to $\sigma_2$ labeled with $b$.*

*The CTL formula $A\mathcal{G}E\mathcal{G}a$ does not hold, as the path $\sigma_1 \to \sigma_2 \to \sigma_2 \to \ldots$ violates the path property $\mathcal{G}E\mathcal{G}a$.*

*However, the CTL formula $A\mathcal{G}E\mathcal{X}E\mathcal{G}a$ holds.*

**Remark 3.4.** Sometimes it is useful to define sub-logics of CTL that allow quantification in a restrictive manner: ACTL stays for the subset of CTL in which existential quantification cannot be expressed and ECTL for the one excluding universal quantification. ACTL ECTL

**Remark 3.5.** Additionally to the temporal operators in Remark 3.3, also quantifiers have an alternative notation:

$$\begin{array}{lll} \exists & \text{for} & E \\ \forall & \text{for} & A \end{array}$$

$\exists$ $\forall$

For example, the formula $A\mathcal{G}E\mathcal{F}\varphi$ can also be written as $\forall\Box\exists\Diamond\varphi$.

### CTL$^*$

The logic CTL$^*$ is an extension of LTL and CTL and allows arbitrary alternation of path quantifiers and temporal operators.

**Definition 3.11 (Syntax of CTL$^*$).** *Assume a set $AP$ of atomic propositions.* CTL$^*$ state formulae *can be built according to the abstract grammar*

$$\psi \ ::= \ a \mid (\psi \wedge \psi) \mid (\neg\psi) \mid (E\varphi)$$

*with $a \in AP$ and where $\varphi$ is a CTL$^*$ path formula.*

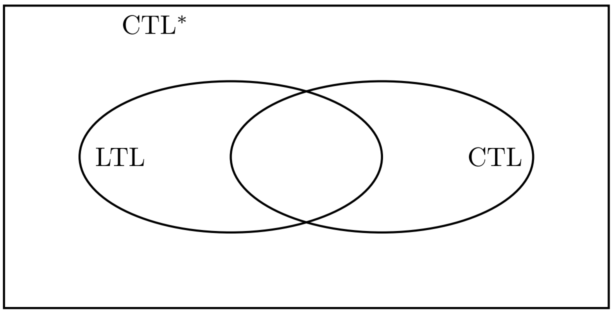


Figure 3.2: The expressiveness of LTL, CTL, and CTL*

CTL* path formulae *are built according to the abstract grammar*

$$\varphi \;\; ::= \;\; \psi \mid (\varphi \wedge \varphi) \mid (\neg\varphi) \mid (\mathcal{X}\varphi) \mid (\varphi\,\mathcal{U}\,\varphi)$$

*where $\psi$ is a CTL* state formula.*

$Form_{CTL^*}$ CTL* formulae *are CTL* state formulae building the set $Form^{AP}_{CTL^*}$ (or short $Form_{CTL^*}$).*

Again, we omit parentheses when it causes no confusion. As in CTL, we can define the "finally" and "globally" operators also for CTL* as syntactic sugar. Note that the universal quantification is not part of the CTL* syntax since $A\varphi$ can be defined as syntactic sugar by $\neg E \neg \varphi$.

**Definition 3.12 (Semantics of CTL*).** *Assume an atomic proposition set $AP$, a labeled state transition system $\mathcal{LSTS} = (\Sigma, Lab, Edge, Init)$ and a state labeling function $L : \Sigma \to 2^{AP}$. CTL* state formulae are evaluated in the context of a state and CTL* path formulae in the context of a path by the satisfaction relation $\models_{CTL^*} \subseteq (\Sigma \cup \Pi) \times Form_{CTL^*}$ (or short $\models$) as follows:*

$\models_{CTL^*}$

$$\begin{array}{lll}
\sigma \models_{CTL^*} a & \textit{iff} & a \in L(\sigma) \\
\sigma \models_{CTL^*} \psi_1 \wedge \psi_2 & \textit{iff} & (\sigma \models_{CTL^*} \psi_1) \wedge (\sigma \models_{CTL^*} \psi_2) \\
\sigma \models_{CTL^*} \neg\psi & \textit{iff} & \sigma \not\models_{CTL^*} \psi \\
\sigma \models_{CTL^*} E\varphi & \textit{iff} & \exists \pi \in \Pi(\sigma).\ \pi \models_{CTL^*} \varphi \\
\\
\pi \models_{CTL^*} \psi & \textit{iff} & \pi(0) \models_{CTL^*} \psi \\
\pi \models_{CTL^*} \varphi_1 \wedge \varphi_2 & \textit{iff} & (\pi \models_{CTL^*} \varphi_1) \wedge (\pi \models_{CTL^*} \varphi_2) \\
\pi \models_{CTL^*} \neg\varphi & \textit{iff} & \pi \not\models_{CTL^*} \varphi \\
\pi \models_{CTL^*} \mathcal{X}\varphi & \textit{iff} & \pi^1 \models_{CTL^*} \varphi \\
\pi \models_{CTL^*} \varphi_1 \mathcal{U}\, \varphi_2 & \textit{iff} & \exists 0 \leq j.\ (\pi^j \models_{CTL^*} \varphi_2) \wedge \forall 0 \leq i < j.\ (\pi^i \models_{CTL^*} \varphi_1)\ .
\end{array}$$

*For $\psi \in Form_{CTL^*}$ we define $\mathcal{LSTS} \models_{CTL^*} \psi$ iff $\sigma_0 \models_{CTL^*} \psi$ for all $\sigma_0 \in Init$.*

### The Relation of LTL, CTL, and CTL*

The logics LTL and CTL are incomparable, and both are included in CTL*, as shown in Figure 3.2. That LTL and CTL are incomparable means, that there are LTL formulae for which no equivalent CTL formulae exist, and vice versa, there are CTL formulae which are not expressible in LTL.

**Example 3.10.**

- *The LTL formula $\mathcal{F}\mathcal{G}a$ is not expressible in CTL.*
- *The CTL formula $A\mathcal{F}A\mathcal{G}a$ is not expressible in LTL.*

There are CTL$^*$ formulae that syntactically does not belong to LTL or to CTL but for that semantically equivalent LTL or CTL formulae can be given. However, CTL$^*$ is more expressive than LTL and CTL together, i.e., there are CTL$^*$ formulae that can be expressed neither in LTL nor in CTL (see Exercise 3.9).

**Example 3.11.** *The CTL$^*$ formula $A\neg\mathcal{G}E\mathcal{F}a$ with $a \in AP$ is syntactically not a CTL formula. However, it can be expressed by the semantically equivalent CTL formula $A\mathcal{F}A\mathcal{G}\neg a$.*

*The CTL$^*$ formula $A\mathcal{G}A\mathcal{F}\mathcal{G}a$ with $a \in AP$ is syntactically not an LTL formula. However, it can be expressed by the semantically equivalent LTL formula $\mathcal{F}\mathcal{G}a$.*

## 3.3 CTL Model Checking for LSTSs

Model checking of discrete systems is not the basic content of this lecture, therefore here we restrict ourselves to the intuition behind explicit[1] CTL model checking for LSTSs, which can handle *finite-state* systems, only. This will be relevant later, as we will build *finite abstractions* of infinite-state systems to be able to apply model checking to them. For more details on model checking we refer to [BK08].

Given an LSTS, an atomic proposition set $AP$, a state labeling function and a CTL (state) formula $\psi_0$, CTL model checking labels the states of the LSTS recursively with the sub-state-formulae of $\psi_0$ inside-out, depending on the type of the subformula:

$a$: The labeling with atomic propositions $a \in AP$ is given by the labeling function.

$\psi_1 \wedge \psi_2$: Given the labelings for $\psi_1$ and $\psi_2$, we label those states with $\psi_1 \wedge \psi_2$ that are labeled with both $\psi_1$ and $\psi_2$.

$\neg\psi$: Given the labeling for $\psi$, we label those states with $\neg\psi$ that are not labeled with $\psi$.

$E\mathcal{X}\psi$: Given the labeling for $\psi$, we label those states with $E\mathcal{X}\psi$ that have a successor state labeled with $\psi$.

$E\psi_1\,\mathcal{U}\,\psi_2$: Given the labeling for $\psi_1$ and $\psi_2$, we

- label all with $\psi_2$ labeled states additionally with $E\psi_1\,\mathcal{U}\,\psi_2$, and
- label those states that have the label $\psi_1$ and have a successor state with the label $E\psi_1\,\mathcal{U}\,\psi_2$ also with $E\psi_1\,\mathcal{U}\,\psi_2$ iteratively until a fixed point is reached, i.e., until no new labels can be added.

$A\mathcal{X}\psi$: Given the labeling for $\psi$, we label those states with $A\mathcal{X}\psi$ whose successor states are all labeled with $\psi$.

[1] Explicit model checking is based on the enumeration of states, in contrast to symbolic model checking using a symbolic state representation like, e.g., binary decision diagrams (BDDs).

$A\psi_1\,\mathcal{U}\,\psi_2$: Given the labeling for $\psi_1$ and $\psi_2$, we

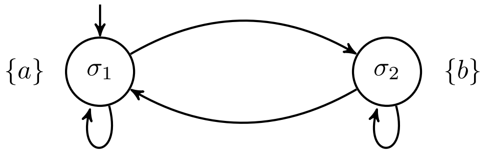
- label all with $\psi_2$ labeled states additionally with $A\psi_1\,\mathcal{U}\,\psi_2$, and
- label those states that have the label $\psi_1$ and all of their successor states have the label $A\psi_1\,\mathcal{U}\,\psi_2$ also with $A\psi_1\,\mathcal{U}\,\psi_2$ iteratively until a fixed point is reached.

The formula $\psi_0$ is satisfied by the LSTS iff after termination of the procedure the initial state is labeled with $\psi_0$.

Since $\psi_0$ has only a finite number of sub-formulae and since there is only a finite number of states that can be labeled in the iterative cases, the procedure always terminates. Note that this model checking approach would not be complete, i.e., it would not terminate, for infinite-state systems.

**Theorem 3.1 (Time complexity of CTL model checking for LSTS [BK08]).** *Assume an LSTS $\mathcal{LSTS}$ with $N$ states and $K$ edges, an atomic proposition set $AP$, a state labeling function and a CTL formula $\psi$ with $M$ subformulae. Then the problem to decide whether $\mathcal{LSTS} \models_{CTL} \psi$ holds can be answered in time $\mathcal{O}((N+K)\cdot M)$.*

**Example 3.12.** *Assume again the LSTS from Example 3.7:*



*In Example 3.9 we stated that this LSTS satisfies the CTL formula $A\mathcal{G}E\mathcal{X}E\mathcal{G}a$. Now we can prove this fact using model checking.*

*First we replace the syntactic sugar of the "globally" operator by its definition using*

$$\begin{array}{rcl} E\mathcal{G}\psi & \leftrightarrow & \neg A\,true\,\mathcal{U}\,\neg\psi \\ A\mathcal{G}\psi & \leftrightarrow & \neg E\,true\,\mathcal{U}\,\neg\psi. \end{array}$$

*This yields*

$$\psi := \neg\left(E\,true\mathcal{U}\,\neg\left(E\mathcal{X}\neg\left(A\,true\mathcal{U}\;(\neg a)\right)\right)\right)\ .$$

*Model checking this property for the given system consists of labeling the states with the following subformulae in this order:*

1. $\psi_1 := \neg a$
2. $\psi_2 := A\,true\mathcal{U}\,\psi_1$
3. $\psi_3 := \neg\psi_2$
4. $\psi_4 := E\mathcal{X}\psi_3$
5. $\psi_5 := \neg\psi_4$
6. $\psi_6 := E\,true\mathcal{U}\,\psi_5$

*7.* $\psi_7 := \neg\psi_6$

*Labeling with the atomic proposition a is given by the labeling function: it holds only in* $\sigma_1$*. For the labeling with the above subformulae we get:*

*1.* $\psi_1 := \neg a$*: We label with* $\psi_1$ *all those states where a does not hold. That means we label* $\sigma_2$ *with* $\psi_1$*.*

*2.* $\psi_2 := A \mathit{true} \mathcal{U} \psi_1$*:*

- *We first label with* $\psi_2$ *all those states where* $\psi_1$ *holds. That means, we label* $\sigma_2$ *with* $\psi_2$*.*
- *Those states that are not yet labeled with* $\psi_2$ *but whose successors are all labeled with* $\psi_2$ *get also labeled with* $\psi_2$*. However, there are no such states.*

*3.* $\psi_3 = \neg\psi_2$*: We label with* $\psi_3$ *all states that are not labeled with* $\psi_2$*. That means, we label* $\sigma_1$ *with* $\psi_3$*.*

*4.* $\psi_4 := E\mathcal{X}\psi_3$*: We label with* $\psi_4$ *all states that have a successor state labeled with* $\psi_3$*. That means, we label both* $\sigma_1$ *and* $\sigma_2$ *with* $\psi_4$*.*

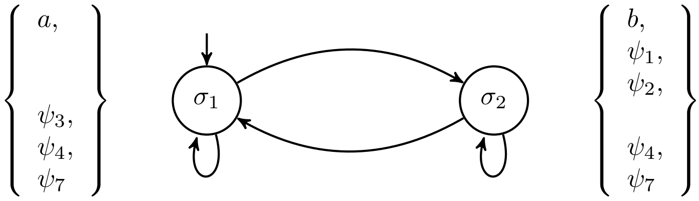
*5.* $\psi_5 := \neg\psi_4$*: Label with* $\psi_5$ *all states that are not labeled with* $\psi_4$*. As both states are labeled with* $\psi_4$*, no states get the label* $\psi_5$ *attached.*

*6.* $\psi_6 := E \mathit{true} \mathcal{U} \psi_5$*:*

- *We label with* $\psi_6$ *all states with the label* $\psi_5$*. However, there are no such states.*
- *We label with* $\psi_6$ *all states that are not yet labeled with* $\psi_6$ *but that have a successor state labeled with* $\psi_6$*. There are no such states.*

*7.* $\psi_7 := \neg\psi_6$*: We label with* $\psi_7$ *all states that are not labeled with* $\psi_6$*. That means, we label both states* $\sigma_1$ *and* $\sigma_2$ *with* $\psi_7$*.*

*The labeling result is as follows:*


$\{a, \psi_3, \psi_4, \psi_7\}$
$\sigma_1$
$\sigma_2$
$\{b, \psi_1, \psi_2, \psi_4, \psi_7\}$

*As the initial state is labeled with* $\psi_7$*, the LSTS satisfies* $\psi_7$*.*

# Exercises

## Modeling Languages

**Exercise 3.1.** Show that the LSTS parallel composition is commutative and associative.

**Exercise 3.2.** Show that the LTS parallel composition is commutative and associative.

**Exercise 3.3.** Construct the parallel composition of the automata for the controller and the gate from Example 3.3.

## Logics

**Exercise 3.4.** Assume $AP = \{a, b\}$. Which of the following formulae are well-formed LTL formulae, i.e., contained in $Form_{LTL}$? Which ones are well-formed CTL formulae, i.e., contained in $Form_{CTL}$?

- $a$
- $a\, \mathcal{U}\, b$
- $E\ a\, \mathcal{U}\, b$
- $a\, \mathcal{U}\ (b\, \mathcal{U}\ c)$
- $E\ a\, \mathcal{U}\ (b\, \mathcal{U}\ c)$
- $a\ \vee A\ \mathcal{G}\ b$
- $A\ \mathcal{F}\ E\ \mathcal{G}\ a$
- $E\ \mathcal{F}\ ((E\ \mathcal{X}a)\ \mathcal{U}\ (A\ \mathcal{G}\ b))$

**Exercise 3.5.** Express the following properties as CTL$^*$ formulae using the atomic propositions from $AP = \{c_1, c_2\}$ with the meaning that $c_i$, $i = 1, 2$ holds whenever process $P_i$ is in its critical section.

- It never happens that both processes $P_1$ and $P_2$ are in their critical sections simultaneously.
- Process $P_1$ will surely enter its critical section.
- For each possible program execution, process $P_1$ will eventually enter its critical section the first time and it happens before $P_2$ does so.
- It might happen that none of the processes ever enter their critical sections.
- Whenever one of the processes enters its critical section it will leave it after a finite number of computation steps.
- Process $P_1$ enters its critical section only finitely many times.
- Starting with $P_1$, the two processes enter their critical sections alternatingly forever.

**Exercise 3.6.** Assume the atomic proposition set $AP = \{send, receive, correct, terminate\}$. Which of the following LTL formulae are safety properties? Which are liveness properties?

- $\mathcal{G}(send \to (\mathcal{F}\ receive))$
- $\mathcal{F}(\mathcal{G}\ \neg send)$
- $\mathcal{G}(send \vee (\mathcal{X}\ send))$
- $\mathcal{F}\ send$
- $\mathcal{G}\ (\neg receive)$
- $send\, \mathcal{U}\ receive)$
- $send\, \mathcal{U}_{weak}\ receive)$
- $\mathcal{G}\ (terminate \to correct)$
- $\mathcal{F}\ terminate$

**Exercise 3.7.** Show that the LTL and the CTL definitions for the "globally" operator are equivalent.

**Exercise 3.8.** Assume an LSTS $\mathcal{LSTS}$, a set $AP$ of atomic propositions and a labeling function $L : \Sigma \to 2^{AP}$. Are the following statement true? Prove or disprove.

1. $\forall \psi \in Form_{LTL}.\ ((\mathcal{LSTS} \models_{LTL} \psi) \quad \leftrightarrow \quad (\mathcal{LSTS} \models_{CTL^*} A\psi))$
2. $\forall \psi \in Form_{CTL}.\ ((\mathcal{LSTS} \models_{CTL} \psi) \quad \leftrightarrow \quad (\mathcal{LSTS} \models_{CTL^*} \psi))$

**Exercise 3.9.** To show that CTL$^*$ is more expressive than LTL and CTL together, give a CTL$^*$ formula that can be expressed neither in LTL nor in CTL.

**Exercise 3.10.** Define a meaningful syntax and semantics for the language ACTL from Remark 3.4.

## CTL Model Checking for LSTS

**Exercise 3.11.** Apply the CTL model checking algorithm to the LSTS from Example 3.9 to decide whether $A\mathcal{G}(a \vee E\mathcal{G}b)$ holds or not.

# Chapter 4

# Discrete-Time Models

Though discrete systems have no continuous components in their model, the real-time behavior of the modeled systems may nonetheless be relevant. Assume a controller executing a program. The program itself can be modeled as a discrete system, however, it may be critical if the program executes too long and the control values arrive too late.

If we want to model time without having a hybrid model, we can use a *discrete-time model*: Time is modeled by discrete time steps, also called *ticks*. Each transition step lasts for exactly one tick. Thus the elapsed time between two actions is always a multiple of a tick.

In order to describe the time behavior of discrete-time systems, the temporal operators of LTL, CTL, and CTL$^*$ can be extended with time bounds. This way we can express not only that some events take place but also *when* they take place in time. However, this extension does not increase the expressive power of the logics, i.e., a formula in the extended logics can be represented with an equivalent formula without the discrete-time extension. This has the effect that we can use model checking for LTL, CTL and CTL$^*$ also for their discrete-time extensions.

Remember that only the temporal operators "next" $\mathcal{X}$ and "until" $\mathcal{U}$ are basic, the remaining ones like "finally" $\mathcal{F}$ and "globally" $\mathcal{G}$ are syntactic sugar.

We extend the "next" operator $\mathcal{X}$ with an upper index. The formula $\mathcal{X}^k\varphi$ with $k \in \mathbb{N}$ denotes that $\varphi$ is true after $k$ steps. This indexed "next" operator does not increase the expressiveness of the logic, as it is syntactic sugar. In LTL it is defined recursively by

$$\mathcal{X}^k\varphi \quad = \quad \begin{cases} \varphi & \textit{if } k = 0 \\ \mathcal{X}\mathcal{X}^{k-1}\varphi & \textit{else.} \end{cases}$$

Thus $\mathcal{X}^k\varphi = \underbrace{\mathcal{X}\ldots\mathcal{X}}_{k}\varphi$ in LTL.

In CTL the quantifiers and temporal operators are alternating. For CTL we define

$$E\mathcal{X}^k\psi \quad = \quad \begin{cases} \psi & \textit{if } k = 0 \\ E\mathcal{X}E\mathcal{X}^{k-1}\psi & \textit{else.} \end{cases}$$

Thus $E\mathcal{X}^k\psi = \underbrace{E\mathcal{X}\ldots E\mathcal{X}}_{k}\psi$. The definition in combination with the universal quantifier $A\mathcal{X}^k\psi$ is analogous.

The extension of the "until" $\mathcal{U}$ operator is similar, but here we allow intervals instead of fixed values for the time bounds. The formula $\varphi_1\ \mathcal{U}^{[k_1,k_2]}\ \varphi_2$ ($k_1, k_2 \in \mathbb{N}$, $k_1 \leq k_2$) states that there exists a $k \in \mathbb{N}$ with $k_1 \leq k \leq k_2$ such that $\varphi_2$ holds in $k$ steps and $\varphi_1$ holds all the time before.

We also allow right-open intervals with $k_2$ being $\infty$, such that we can still represent the original "until" operator by $\varphi_1 \; \mathcal{U}^{[0,\infty)} \; \varphi_2 = \varphi_1 \; \mathcal{U} \; \varphi_2$.
In LTL we define

$$\varphi_1 \; \mathcal{U}^I \; \varphi_2 \quad = \quad \begin{cases} \varphi_1 \; \mathcal{U} \; \varphi_2 & \textit{for } I = [0,\infty) \\ \varphi_2 & \textit{for } I = [0,0] \\ \varphi_1 \wedge \mathcal{X}(\varphi_1 \; \mathcal{U}^{[k_1-1,k_2-1]} \; \varphi_2) & \textit{for } I = [k_1,k_2],\ k_1 > 0 \\ \varphi_2 \vee (\varphi_1 \wedge \mathcal{X}(\varphi_1 \; \mathcal{U}^{[0,k_2-1]} \; \varphi_2)) & \textit{for } I = [k_1,k_2],\ k_1 = 0, k_2 > 0\ . \end{cases}$$

In CTL we define

$$E\psi_1 \; \mathcal{U}^I \; \psi_2 \quad = \quad \begin{cases} E\psi_1 \; \mathcal{U} \; \psi_2 & \textit{for } I = [0,\infty) \\ \psi_2 & \textit{for } I = [0,0] \\ \psi_1 \wedge E\mathcal{X}E(\psi_1 \; \mathcal{U}^{[k_1-1,k_2-1]} \; \psi_2) & \textit{for } I = [k_1,k_2],\ k_1 > 0 \\ \psi_2 \vee (\psi_1 \wedge E\mathcal{X}E(\psi_1 \; \mathcal{U}^{[0,k_2-1]} \; \psi_2)) & \textit{for } I = [k_1,k_2],\ k_1 = 0, k_2 > 0\ . \end{cases}$$

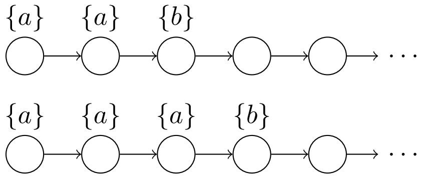
We also write

- $\mathcal{U}^{\leq k}$ instead of $\mathcal{U}^{[0,k]}$,
- $\mathcal{U}^{\geq k}$ for $\mathcal{U}^{[k,\infty]}$,
- $\mathcal{U}^{=k}$ for $\mathcal{U}^{[k,k]}$, and
- $\mathcal{U}$ for $\mathcal{U}^{[0,\infty]}$.

**Example 4.1.** *The discrete-time LTL formula $a \; \mathcal{U}^{[2,3]} \; b$ is defined as*

$$a \wedge \mathcal{X}(a \wedge \mathcal{X}(b \vee (a \wedge \mathcal{X}b))).$$

*It is satisfied by paths of the following form:*



As the discrete-time temporal operators are defined as syntactic sugar, model checking can be applied to check the validity of discrete-time temporal formulae for labeled state transition systems [Kat99, CGP01].

# Exercises

**Exercise 4.1.** For non-empty intervals $I \subseteq \mathbb{N}$ of the form $[k_1, k_2]$ or $[k_1, \infty)$ ($k_1, k_2 \in \mathbb{N}$, $k_1 \leq k_2$), define the discrete-time LTL operators $\mathcal{F}^I\varphi$ and $\mathcal{G}^I\varphi$ as syntactic sugar.

**Exercise 4.2.** For non-empty intervals $I \subseteq \mathbb{N}$ of the form $[k_1, k_2]$ or $[k_1, \infty)$ ($k_1, k_2 \in \mathbb{N}$, $k_1 \leq k_2$), define the discrete-time CTL operators $A\mathcal{F}^I\varphi$, $E\mathcal{F}^I\varphi$, $A\mathcal{G}^I\varphi$ and $E\mathcal{G}^I\varphi$ as syntactic sugar.

# Chapter 5

# Timed Automata

The popular modeling formalism of *timed automata* combines labeled transition system models with a notion of *time* as the only continuous component. Its success is based on two main facts: Firstly, this model class, despite its rather weak expressiveness, already allows to model a wide range of real-time systems. Secondly, the model checking problem for safety and liveness properties of timed automata is still efficiently decidable. Driven by both academic and industrial interests, a lot of effort was put into tool support. Uppaal is one of the most widely used tools for model checking timed automata.

In this chapter we first introduce *timed automata* in Section 5.1. In Section 5.2 we extend the logic CTL with continuous-time aspects, resulting in the logic *timed CTL* (*TCTL*). In this book we restrict ourselves to the introduction of TCTL. Another popular timed temporal logic is, e.g., metric LTL (MTL). We discuss *model checking* TCTL properties of timed automata in Section 5.3. For further reading on timed automata and its model checking algorithm we refer to [BK08].

## 5.1 Syntax and Semantics

A timed automaton has a finite number of *clocks* as variables. A clock measures the time, i.e., it continuously evolves at rate 1. The values of the clocks can only be accessed in a limited way. For read access, the only fact we can observe about a clock value is the result of a comparison of its value with a constant. Such comparisons can be formulated by *clock constraints.* For write access, clocks can only be *reset*, i.e., their values can only be set to 0.

**Definition 5.1 (Syntax of clock constraints).** Clock constraints *over a finite set $\mathcal{C}$ of clocks can be built using the following abstract grammar:*

$$g \quad ::= \quad x < c \mid x \leq c \mid x > c \mid x \geq c \mid g \wedge g$$

*where $c \in \mathbb{N}$*[1] *and $x \in \mathcal{C}$.*
*Clock constraints which are not a conjunction are called* atomic. *The set of atomic clock constraints over a set $\mathcal{C}$ of clocks is denoted by $ACC_{\mathcal{C}}$. The set of all clock constraints over $\mathcal{C}$ is referred to as $CC_{\mathcal{C}}$.*

Clock constraints are evaluated in the context of a *valuation* $\nu : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$ assigning non-negative real values to clocks. We use $V_{\mathcal{C}}$ (or short $V$) for the set of all valuations.

[1] We could also allow $c \in \mathbb{Q}$.

$\models_{CC}$

**Definition 5.2 (Semantics of clock constraints).** *The semantics of clock constraints over a finite set $\mathcal{C}$ of clocks is given by the relation $\models_{CC} \subseteq V \times CC_{\mathcal{C}}$ (or short $\models$) defined as follows:*

$$\begin{array}{lll}
\nu \models_{CC} x < c & \textit{iff} & \nu(x) < c \\
\nu \models_{CC} x \leq c & \textit{iff} & \nu(x) \leq c \\
\nu \models_{CC} x > c & \textit{iff} & \nu(x) > c \\
\nu \models_{CC} x \geq c & \textit{iff} & \nu(x) \geq c \\
\nu \models_{CC} g_1 \wedge g_2 & \textit{iff} & (\nu \models_{CC} g_1) \wedge (\nu \models_{CC} g_2) \ .
\end{array}$$

For the sake of readability we also use notations like

$$\textit{true}, \quad x \in [c_1, c_2), \quad c_1 \leq x < c_2, \quad x = c, \ldots$$

with the expected meaning. E.g., $x = c$ can be defined using $x \geq c \wedge x \leq c$.

Based on its semantics, a clock constraint $g \in CC_{\mathcal{C}}$ can also be seen as the set $\{\nu \in V \mid \nu \models g\}$ of all valuations that satisfy $g$.

**Remark 5.1.** Note that the syntax of clock constraints allows conjunction but no negation, assuring that the sets defined by clock constraints are *convex*. This has the big advantage that, when we start time progress with a valuation $\nu \in V$ satisfying a clock constraint $g \in CC_{\mathcal{C}}$ then, since time progress is linear, when a valuation $\nu + t$ after some time elapse $t \in \mathbb{R}_{\geq 0}$ still satisfies the clock constraint $g$ then we know that all the valuations inbetween also satisfied $g$, i.e., $\nu + t' \models g$ for all $0 \leq t' \leq t$.

As mentioned above, write access to clocks is restricted to resetting their values to 0.

*reset*$(C)$

**Definition 5.3 (Syntax of clock reset).** *Given a finite set $\mathcal{C}$ of clocks, a* clock reset *is an expression of the form reset$(C)$ with $C \subseteq \mathcal{C}$.*

Sometimes we also write $\textit{reset}(x_1, \ldots, x_n)$ instead of $\textit{reset}(\{x_1, \ldots, x_n\})$.

Also the semantics of a clock reset is given in the context of a valuation. Semantically, a clock reset $\textit{reset}(C)$ denotes that the values of all clocks in $C$ get reset to 0, and the values of all other clocks from $\mathcal{C} \backslash C$ remain unchanged.

**Definition 5.4 (Semantics of clock reset).** *Let $\mathcal{C}$ be a finite set of clocks and $C \subseteq \mathcal{C}$. The result of reset$(C)$ applied to a valuation $\nu \in V$ is given by the valuation satisfying*

*reset*$(C)$ *in* $\nu$

$$(\textit{reset}(C) \ \textit{in} \ \nu)(x) = \begin{cases} 0 & \textit{if } x \in C \\ \nu(x) & \textit{otherwise} \end{cases}$$

*for all $x \in \mathcal{C}$.*

The following notation formalizes time delay.

$\nu + t$

**Definition 5.5.** *For all valuations $\nu \in V$ and constants $t \in \mathbb{R}_{\geq 0}$ we define the valuation $\nu + t$ by $(\nu + t)(x) = \nu(x) + t$ for all $x \in \mathcal{C}$.*

**Example 5.1 (Clock access).** *Assume a clock set $\mathcal{C} = \{x, y\}$ and a valuation $\nu : \mathcal{C} \to \mathbb{R}_{\geq 0}$ with $\nu(x) = 2$ and $\nu(y) = 3$. Then*

- *$\nu + 9$ assigns 11 to $x$ and 12 to $y$,*
- *$\textit{reset}(x)$ in $(\nu + 9)$ assigns 0 to $x$ and 12 to $y$,*
- *$(\textit{reset}(x)$ in $\nu) + 9$ assigns 9 to $x$ and 12 to $y$,*
- *$\textit{reset}(x)$ in $(\textit{reset}(y)$ in $\nu)$ assigns 0 to both $x$ and $y$, and*
- *$\textit{reset}(x, y)$ in $\nu$ assigns 0 to both $x$ and $y$.*

Next we give the definition of timed automata. These models have an LTS component with a rectricted syntax for discrete steps, allowing only clock constraints and clock resets in the definition of the transition relation. This discrete model is extended by introducing time as a continuous quantity: while the control stays in a location, time ellapses and the values of the clocks increase continuously. The main differences to LTS models are the following:

- The variable set of a timed automaton is denoted by $\mathcal{C}$ instead of $Var$ to express the fact that all variables of a timed automaton are *clocks.* assign the value 0 to all clocks. A *state* of a timed automaton is a location-valuation pair $(l, \nu) \in Loc \times V_{\mathcal{C}} = \Sigma$, storing the location of the timed automaton in that the control currently stays together with the current values of the clocks.
- In order to restrict the transition relation of the discrete edges to the less powerful clock access, we use enabling conditions in form of clock constraints combined with reset sets in place of general transition relations. Given a pair $(g, C) \in CC_{\mathcal{C}} \times 2^{\mathcal{C}}$ of a clock constraint $g$ and a reset set $C$, the corresponding *transition relation* $\mu \subseteq V^2$ is given by

  $$\mu = \{(\nu, \nu') \in V^2 \mid \nu \models g \wedge \nu' = reset(C) \ in \ \nu\} \ .$$

  Thus edges have the form $(\ell, a, (g, C), \ell') \in Loc \times Lab \times (CC_{\mathcal{C}} \times 2^{\mathcal{C}}) \times Loc)$.
- As long as the control stays in a location, the values of all clocks *evolve* with the derivative 1. That means, when a location is entered with a valuation $\nu$, after $t$ time the valuation will be $\nu + t$.
- The locations can be annotated with *invariants.* Control may stay in a location only as long as the invariant of the location is not violated. Invariants allow to enforce discrete transitions; without invariants, the control could stay in a location forever. Similarly to the guards of the discrete transitions, also invariants are defined by clock constraints.
- There is a further difference between LTS and timed automata regarding the parallel composition. For LTS the parallel composition supports shared variables accessible by different components. However, allowing shared variables in the timed automata composition would lead to some complications, which we do not discuss here. Instead, we restrict the composition of timed automata to components having disjoint variable sets. Note that when excluding shared variables, the only way of communication is label synchronization. Thus the definition of the controlled variable sets $Con$ and also the $\tau$-transitions get superfluous.

**Definition 5.6 (Syntax of timed automata).** *A* timed automaton *is a tuple* $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$ *with*

- *$Loc$ is a finite set of* locations*,*
- *$\mathcal{C}$ is a finite set of real-valued variables called* clocks*,*
- *$Lab$ is a finite set of* synchronization labels*,*
- *$Edge \subseteq Loc \times Lab \times (CC_{\mathcal{C}} \times 2^{\mathcal{C}}) \times Loc$ is a finite set of* edges*,*
- *$Inv : Loc \to CC_{\mathcal{C}}$ is a function assigning an* invariant *to each location, and*
- *$Init \subseteq Loc \times V_{\mathcal{C}} = \Sigma$ a set of* initial states *with $\nu(x) = 0$ for all $(\ell, \nu) \in Init$ and each $x \in \mathcal{C}$.*

To simplify the formalisms, we extend the notations for valuations to states and use $\sigma \models_{CC} g$ for a state $\sigma = (\ell, \nu)$ to express that $\nu \models_{CC} g$. Similarly, for $\sigma = (\ell, \nu)$ we also write $reset(x) \ in \ \sigma$ to denote $(\ell, reset(x) \ in \ \nu)$.

**Definition 5.7 (Semantics of timed automata).** *The* operational semantics *of a timed automaton $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$ is given by the following two rules:*

$$\frac{(\ell, a, (g, C), \ell') \in Edge \quad \nu \models g \quad \nu' = reset(C) \text{ in } \nu \quad \nu' \models Inv(\ell')}{(\ell, \nu) \xrightarrow{a} (\ell', \nu')} \; \texttt{Rule}_{\texttt{discrete}}$$

$$\frac{t \in \mathbb{R}_{\geq 0} \quad \nu' = \nu + t \quad \nu' \models Inv(\ell)}{(\ell, \nu) \xrightarrow{t} (\ell, \nu')} \; \texttt{Rule}_{\texttt{time}} \; .$$

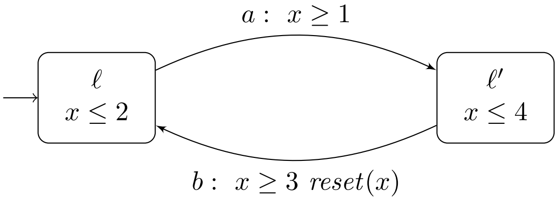
*We write $\sigma \to \sigma'$ instead of $\sigma \xrightarrow{a} \sigma'$ or $\sigma \xrightarrow{t} \sigma'$ when the type of the step is not of interest.*
*A* run *(or* path *or* execution*) of $\mathcal{T}$ is an infinite sequence $\sigma_0 \to \sigma_1 \to \sigma_2 \ldots$ with $\sigma_i \in \Sigma$ and $\sigma_0 = (\ell_0, \nu_0) \in Inv(\ell_0)$; if additionally $\sigma_0 \in Init$ then we call $\pi$ an* initial path.
*We use $\Pi_{\mathcal{T}}(\sigma)$ (or short $\Pi(\sigma)$) to denote the set of all paths of $\mathcal{T}$ starting in $\sigma \in \Sigma$, and define $\Pi_{\mathcal{T}} = \bigcup_{\sigma \in \Sigma} \Pi_{\mathcal{T}}(\sigma)$ (or short $\Pi$). A state is* reachable *if there is an initial path leading to it; we write $Reach_{\mathcal{T}}$ (or short Reach) for the set of reachable states of $\mathcal{T}$.*

Note that, since the invariants are convex sets, it is enough to require that they hold *after* each time step, and we do not need the requirement that they hold *during* the whole period of a time step. Together with the requirement that the starting states of paths satisfy the corresponding invariants, we get by induction that the invariants hold on all paths at each time point.

Again, the semantics of a timed automaton induces an LSTS for its (in general uncountable) state space. As in the case of discrete systems, also timed automata can be augmented by a *labeling function*. However, since the state space is now uncountable, we attach propositions to the locations (instead of the states) by a labeling function $L : Loc \to 2^{AP}$ where $AP$ denotes the set of atomic propositions. To simplify the notation, we overload the labeling function defining $L : \Sigma \to 2^{AP}$ with $L((\ell, \nu)) = L(\ell)$.

Timed automata are often represented graphically, where non-synchronizing labels, trivial conditions and empty reset sets are skipped. As all clocks evolve with derivative 1 we do not represent the time behavior in the graphs.

**Example 5.2.** *The graphical representation*



*denotes the timed automaton $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$ with*

- $Loc = \{\ell, \ell'\}$
- $\mathcal{C} = \{x\}$,
- $Lab = \{a, b\}$,
- $Edge = \{(\ell,\ a,\ (x \geq 1,\ \emptyset),\ \ell'),\ (\ell',\ b,\ (x \geq 3,\ \{x\}),\ l)\}$,
- $Inv(\ell) = x \leq 2$, $Inv(\ell') = x \leq 4$,
- $Init = \{(\ell, \nu_0)\}$ *with* $\nu_0(x) = 0$.

**Definition 5.8 (Parallel composition of timed automata).** *Let $\mathcal{T}_1 = (Loc_1, \mathcal{C}_1, Lab_1, Edge_1, Inv_1, Init_1)$ and $\mathcal{T}_2 = (Loc_2, \mathcal{C}_2, Lab_2, Edge_2, Inv_2, Init_2)$ two timed automata with $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.*

*The* parallel composition $\mathcal{T}_1||\mathcal{T}_2$ *is a timed automaton* $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$ *with valuations* $\nu : \mathcal{C} \to \mathbb{R}_{\geq 0}$*, valuation set* $V$*, and states* $\Sigma = Loc \times V$*, where*

- $Loc = Loc_1 \times Loc_2$
- $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$
- $Lab = Lab_1 \cup Lab_2$
- $Inv((\ell_1, \ell_2)) = Inv_1(\ell_1) \wedge Inv_2(\ell_2)$ *for all* $(\ell_1, \ell_2) \in Loc$
- $Init = \{((\ell_1, \ell_2), \nu) \in \Sigma \mid (\ell_1, \nu) \in Init_1 \wedge (\ell_2, \nu) \in Init_2\}$
- $Edge =$

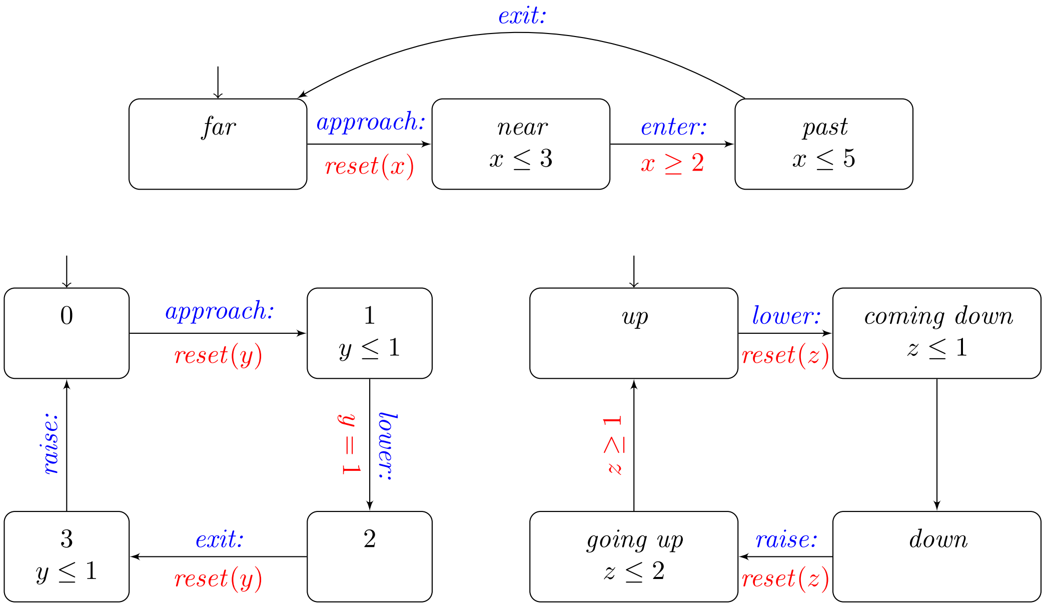
$$\begin{array}{rcl}
\{((\ell_1, \ell_2), a, (g_1 \wedge g_2, \mathcal{C}_1 \cup \mathcal{C}_2), (\ell_1', \ell_2')) & \mid & (\ell_1, a, (g_1, \mathcal{C}_1), \ell_1') \in Edge_1 \wedge \\
 & & (\ell_2, a, (g_2, \mathcal{C}_2), \ell_2') \in Edge_2\} \\
\{((\ell_1, \ell_2), a, (g, \mathcal{C}), (\ell_1', \ell_2)) & \mid & (\ell_1, a, (g, \mathcal{C}), \ell_1') \in Edge_1 \wedge a \notin Lab_2\} \\
\{((\ell_1, \ell_2), a, (g, \mathcal{C}), (\ell_1, \ell_2')) & \mid & (\ell_2, a, (g, \mathcal{C}), \ell_2') \in Edge_2 \wedge a \notin Lab_1\}\ .
\end{array}$$

To illustrate the parallel composition, we extend our previous LSTS railroad crossing model to a timed automaton model.

**Example 5.3 (Railroad crossing).** *We extend Example 3.3 with real-time behavior as follows:*

- *After the train triggers the "approach" signal it reaches the gate between* 2 *and* 3 *minutes. It passes the track between the "approach" and the "exit" sensors within* 5 *minutes. The timed automaton* $\mathcal{H}_{Train}$ *modeling the train has a clock* $x$ *which is a control variable in each location.*
- *After receiving an "approach" signal, the controller delays* 1 *minute before it sends a "lower" signal to the gate. After receiving an "exit" signal it notifies the gate by emitting a "raise" signal with a delay of at most one minute. The timed automaton model* $\mathcal{H}_{Controller}$ *of the controller has a clock* $y$ *being a control variable in each location.*
- *The gate needs at most one minute to be lowered and between one and two minutes to be raised. The timed automaton model* $\mathcal{H}_{Gate}$ *has its own clock* $z$*, being a control variable in each location.*

*Adapting the syntax of timed automata we get the following graphical representation:*



### 5.1.1 Continuous-Time Phenomena

Similarly to LTS, the semantics of a timed automaton induces a LSTS. Each path in the induced LSTS corresponds to a possible system behavior. However, some of the paths may model unrealistic behavior.

**Time convergence:** There are syntactically inavoidable paths of timed automata along which time converges, i.e., time never evolves beyond some value. For example, the timed automaton from Example 5.2 has a path

$$(\ell,\nu_1) \xrightarrow{1} (\ell,\nu_2) \xrightarrow{1/2} (\ell,\nu_3) \xrightarrow{1/4} (\ell,\nu_4) \xrightarrow{1/8} \ldots$$

starting in the initial state and executing time steps with durations converging to 0. The time duration $\sum_{i=1}^{n} \frac{1}{i}$ converges to 2 with path length $n \rightarrow \infty$. Such a path is called *time-convergent*. Paths that are not time-convergent are called *time-divergent*.

Time-convergent paths are not realizable, but are unavoidable in the modeling. We will explicitly exclude such paths in the semantics of the logic TCTL for the property specification.

**Time lock:** There could be states in the LSTS of a timed automaton from which all paths are time-convergent, such that there is no possibility that time progresses forever. Such states do not allow time divergence, and are therefore called *time locks*. Timed automata without time locks are called *time-lock free*. Time locks are modeling flaws, i.e., they can be avoided by appropriate modeling.

**Zeno paths:** Paths along which infinitely many discrete steps are performed in a finite amount of time are called *Zeno* paths. Note that all Zeno paths are time-convergent. Zeno paths are not realizable, as they would require infinitely fast processors. They are also modeling flaws and can be avoided by careful modeling.

Next we formalize the above properties.

**Definition 5.9 (Time convergence, time lock, Zeno paths).** *For a timed automaton $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$ we define the time duration of a step by the function $ExecTime : (Lab \cup \mathbb{R}_{\geq 0}) \to \mathbb{R}_{\geq 0}$ with* $ExecTime$

$$ExecTime(\alpha) \quad = \quad \begin{cases} 0 & \text{if } \alpha \in Lab \\ \alpha & \text{if } \alpha \in \mathbb{R}_{\geq 0} \end{cases} .$$

*The time duration of an infinite path $\pi = \sigma_0 \overset{\tau_0}{\to} \sigma_1 \overset{\tau_1}{\to} \sigma_2 \overset{\tau_2}{\to} \dots$ of $\mathcal{T}$ is defined by the (overloaded) function*

$$ExecTime(\pi) \quad = \quad \sum_{i=0}^{\infty} ExecTime(\tau_i).$$

- *An infinite path $\pi \in \Pi$ is said to be* time-divergent *if $ExecTime(\pi) = \infty$, and* time-convergent *otherwise.*
  *For a state $\sigma \in \Sigma$ we define $\Pi_{div}(\sigma) \subseteq \Pi(\sigma)$ to be the set of time-divergent infinite paths starting in $\sigma$, and $\Pi_{div} = \bigcup_{\sigma\in\Sigma} \Pi_{div}(\sigma)$.* $\Pi_{div}(\sigma), \Pi_{div}$
- *A state $\sigma \in \Sigma$ contains a* timelock *iff $\Pi_{div}(\sigma) = \emptyset$. A timed automaton is said to be* timelock-free *if none of its reachable states contains a timelock.*
- *An infinite path $\pi \in \Pi$ is said to be* Zeno *if it is time-convergent and infinitely many discrete actions are executed within $\pi$. The timed automaton $\mathcal{T}$ is said to be* non-Zeno *if it has no Zeno paths.*

As mentioned above, Zeno paths a modeling flows. To check whether a timed automaton is non-Zeno is algorithmically difficult. However, there is a sufficient (but not necessary) condition which is simple to check.

**Theorem 5.1 (Sufficient condition for non-Zenoness).** *Assume a timed automaton $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$ such that for each sequence of edges*

$$\ell_0 \xrightarrow{\alpha_1:g_1,C_1} \ell_1 \xrightarrow{\alpha_2:g_2,C_2} \ell_2 \dots \xrightarrow{\alpha_n:g_n,C_n} \ell_n = \ell_0$$

*in $\mathcal{T}$ there exists a clock $x \in \mathcal{C}$ such that*

1. *$x \in C_i$ for some $0 < i \leq n$ and*
2. *for all valuations $\nu \in V$ there exists a $c \in \mathbb{N}_{>0}$ such that*

   $$\nu(x) < c \quad \to \quad (\nu \not\models g_j \text{ or } \nu \not\models Inv(\ell_j))$$

   *for some $0 < j \leq n$.*

*Then $\mathcal{T}$ is non-Zeno.*

## 5.2 Timed Computation Tree Logic (TCTL)

Timed automata often model real-time systems that are *time-critical* in the sense that for their correct functioning certain events must occur within some time bounds. For example, in case of an accident the airbag of a car must react within very tight time limits. Also other controller are supposed to support control values within some predefined time bounds.

The untimed logics of the previous section are not yet able to argue about such time constraints. In this section we extend them for this purpose. Thereby we restrict ourselves to the extension of CTL to *timed CTL* (*TCTL*). The extensions of LTL and CTL$^*$ are analogous.

The main differences between CTL and TCTL are as follows:

- For discrete systems we used an atomic proposition set and a labeling function to assign atomic propositions to states. Besides such atomic propositions, for timed automata we also want to argue about clock values in form of atomic clock constraints. Therefore, both atomic propositions and atomic clock constraints are atomic TCTL state formulae.
- Since timed automata model continuous time, there is no "next" operator in TCTL.
- Remember that a CTL "until" formula $\psi_1 \, \mathcal{U} \, \psi_2$ is satisfied by a path if $\psi_2$ is satisfied by a state somewhere on the path, and $\psi_1$ holds in all the states before. In TCTL, the "until" operator of CTL gets indexed with a time interval. TCTL "bounded until" formulae have the form $\psi_1 \, \mathcal{U}^{[t_1,t_2]} \, \psi_2$, where the time interval $[t_1, t_2]$ puts a restriction *when* $\psi_2$ gets valid. A path satisfies the formula $\psi_1 \, \mathcal{U}^{[t_1,t_2]} \, \psi_2$ if, when measuring the time from the beginning of the path, $\psi_2$ is valid at a time point $t \in [t_1, t_2]$, and $\psi_1 \vee \psi_2$ holds all the time before. (Note that we do not require $\psi_1$ to hold all the time before, but only the weaker statement $\psi_1 \vee \psi_2$.)
- There is a difference between the CTL and the TCTL semantics of quantification over paths. CTL quantification ranges over all paths. However, timed automata have time-convergent paths that cannot be excluded by modeling. Since those paths are not realistic, they are not considered in the TCTL semantics. Therefore, TCLT quantification ranges over time-divergent paths, only.

**Definition 5.10 (Syntax of TCTL).** *TCTL* state formulae *over a set AP of atomic propositions and a set $\mathcal{C}$ of clocks can be built according to the abstract grammar*

$$\psi \quad ::= \quad a \mid g \mid (\psi \wedge \psi) \mid (\neg \psi) \mid (E\varphi) \mid (A\varphi)$$

*with $a \in AP$, $g \in ACC_{\mathcal{C}}$, and $\varphi$ are TCTL path formulae. TCTL* path formulae *are built according to the abstract grammar*

$$\varphi \quad ::= \quad \psi \, \mathcal{U}^J \, \psi$$

*with $J \subseteq \mathbb{R}_{\geq 0}$ is an (open, half-open or closed) interval with integer bounds (open right bound may be $\infty$), and where $\psi$ are TCTL state formulae.* TCTL formulae *are TCTL state formulae.*

Similarly to CTL, we introduce further operators as syntactic sugar. Besides the "finally" and "globally" operators, we consider TCTL formulae with intervals $[0, \infty)$ as CTL formulae.

$$\begin{array}{lcl}
\mathcal{F}^J \psi & := & \mathit{true} \, \mathcal{U}^J \, \psi \\
E\mathcal{G}^J \psi & := & \neg A \mathcal{F}^J \neg \psi \\
A\mathcal{G}^J \psi & := & \neg E \mathcal{F}^J \neg \psi \\
\psi_1 \, \mathcal{U} \, \psi_1 & := & \psi_1 \, \mathcal{U}^{[0,\infty)} \, \psi_2 \\
\mathcal{F}\psi & := & \mathcal{F}^{[0,\infty)} \psi \\
\mathcal{G}\psi & := & \mathcal{G}^{[0,\infty)} \psi
\end{array}$$

For the time bounds on temporal operators, we sometimes write $\leq c$, $< c$, ... instead of the intervals $[0, c]$, $[0, c)$, ....

**Definition 5.11 (Semantics of TCTL).** *Let $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$ be a timed automaton, $AP$ a set of atomic propositions, and $L : Loc \to 2^{AP}$ a state labeling function. The satisfaction relation $\models_{TCTL} \subseteq (\Sigma \cup \Pi_{\mathcal{T}}) \times Form_{TCTL}$ (or short $\models$) evaluates TCTL state and path formulae as follows:*

$$\begin{array}{lll}
\sigma & \models_{TCTL} true & \\
\sigma & \models_{TCTL} a & \textit{iff} \quad a \in L(\sigma) \\
\sigma & \models_{TCTL} g & \textit{iff} \quad \sigma \models_{CC} g \\
\sigma & \models_{TCTL} \neg\psi & \textit{iff} \quad \sigma \not\models_{TCTL} \psi \\
\sigma & \models_{TCTL} \psi_1 \wedge \psi_2 & \textit{iff} \quad \sigma \models_{TCTL} \psi_1 \textit{ and } \sigma \models_{TCTL} \psi_2 \\
 & & \\
\sigma & \models_{TCTL} E\varphi & \textit{iff} \quad \pi \models_{TCTL} \varphi \textit{ for some } \pi \in \Pi_{div}(\sigma) \\
\sigma & \models_{TCTL} A\varphi & \textit{iff} \quad \pi \models_{TCTL} \varphi \textit{ for all } \pi \in \Pi_{div}(\sigma)
\end{array}$$

*where $\sigma \in \Sigma$, $a \in AP$, $g \in ACC(\mathcal{C})$, $\psi$, $\psi_1$ and $\psi_2$ are TCTL state formulae, and $\varphi$ is a TCTL path formula.*
*For an infinite path $\pi = \sigma_0 \overset{\alpha_0}{\to} \sigma_1 \overset{\alpha_1}{\to} \sigma_2 \overset{\alpha_2}{\to} \ldots \in \Pi_{div}$ let $d_i = ExecTime(\alpha_i)$. The satisfaction relation for bounded until formulae is defined by*

$$\begin{array}{lll}
\pi \models_{TCTL} \psi_1 \, \mathcal{U}^J \, \psi_2 & \textit{iff} & \textit{there is an } i \geq 0 \textit{ such that } \sigma_i + d \models_{TCTL} \psi_2 \\
 & & \textit{for some } d \in [0, d_i] \textit{ with } (\sum_{k=0}^{i-1} d_k) + d \in J \\
 & & \textit{and for all } j \leq i \textit{ it holds that } \sigma_j + d' \models_{TCTL} \psi_1 \\
 & & \textit{for any } d' \in [0, d_j] \textit{ with either } j < i \textit{ or } d' < d \,.
\end{array}$$

*We define* Sat

$$Sat(\psi) = \{\sigma \in \Sigma \mid \sigma \models_{TCTL} \psi\}$$

*and*

$$\mathcal{T} \models_{TCTL} \psi \textit{ iff } \forall \sigma = (\ell, \nu) \in Init \cap Inv(\ell).\ \sigma \models_{TCTL} \psi.$$

Note that TCLT quantification ranges over time-divergent paths, only.

**Remark 5.2.** The TCTL semantics introduced above is the so-called *continuous* semantics. There is another interpretation of TCTL formulae based on a *pointwise* semantics, the main difference being that along a path $\pi = \sigma_0 \to \sigma_1 \to \ldots$ only the states $\sigma_i$ are considered in the satisfaction relation but not the other states visited during time steps.

There is also another established variant of the above-defined continuous TCTL semantics, differing in the meaning of the bounded until formula $\psi_1 \, \mathcal{U}^J \, \psi_2$: instead of $\psi_1$ the weaker requirement $\psi_1 \vee \psi_2$ must hold before the time point of $\psi_2$.

$$\begin{array}{lll}
\pi \models_{TCTL} \psi_1 \, \mathcal{U}^J \, \psi_2 & \textit{iff} & \textit{there is an } i \geq 0 \textit{ such that } \sigma_i + d \models_{TCTL} \psi_2 \\
 & & \textit{for some } d \in [0, d_i] \textit{ with } (\sum_{k=0}^{i-1} d_k) + d \in J \\
 & & \textit{and for all } j \leq i \textit{ it holds that } \sigma_j + d' \models_{TCTL} \psi_1 \vee \psi_2 \\
 & & \textit{for any } d' \in [0, d_j] \textit{ with either } j < i \textit{ or } d' < d \,.
\end{array}$$

## 5.3 Model Checking TCTL for Timed Automata

After introducing timed automata and the logic TCTL to define properties of timed automata, in this section we give a model checking algorithm to check whether a TCTL formula holds for a given timed automaton. The main problem for model checking TCTL for timed automata lies in the infinite state space. We use abstraction to solve this problem.

The basic structure of the model checking algorithm is as follows:

Input: Non-Zeno timed automaton $\mathcal{T}$ with clock set $\mathcal{C}$,
a labeling function $L$ over a set of atomic propositions $AP$, and
a TCTL formula $\psi$ over $AP$ and $\mathcal{C}$

Output: The answer to the question whether $\mathcal{T} \models_{TCTL} \psi$

1. Eliminate the timing parameters from $\psi$, resulting in a formula $\hat{\psi}$ which contains atomic clock constraints but *no intervals* on the temporal operators. If we see atomic clock constraints as atomic propositions then $\hat{\psi}$ is a CTL formula.
2. Make a finite abstraction of the state space, with the abstract states called *regions*.
3. Construct an abstract finite transition system $\mathcal{RTS}$ (*region transition system*) with regions as abstract states, and label the regions with atomic propositions and atomic clock constraints. We have $\mathcal{T} \models_{TCTL} \psi$ iff $\mathcal{RTS} \models_{CTL} \hat{\psi}$.
4. Apply *CTL model checking* to check whether $\mathcal{RTS} \models_{CTL} \hat{\psi}$.
5. *Return* the result of the CTL model checking.

Assume in the following an input for the algorithm in form of a timed automaton $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$, a set of atomic propositions $AP$, a labeling function $L : Loc \to 2^{AP}$, and a TCTL formula $\psi$ over $AP$ and $\mathcal{C}$.

### 5.3.1 Eliminating Timing Parameters

Let $\mathcal{T}' = \mathcal{T} \oplus z$ result from $\mathcal{T}$ by adding a fresh clock $z$ which never gets reset. We use this auxiliary clock to measure the time from the beginning of a path and express the time bound of a bounded until as atomic clock constraint. For any state $\sigma$ of $\mathcal{T}$ it holds that

$$\begin{array}{lll} \sigma \models_{TCTL} E(\psi_1\, \mathcal{U}^J\, \psi_2) & \textit{iff} & \textit{reset}(z)\ \textit{in}\ \sigma \models_{TCTL} E\psi_1\, \mathcal{U}\ \ ((z \in J) \wedge \psi_2) \\ \sigma \models_{TCTL} A(\psi_1\, \mathcal{U}^J\, \psi_2) & \textit{iff} & \textit{reset}(z)\ \textit{in}\ \sigma \models_{TCTL} A\psi_1\, \mathcal{U}\ \ ((z \in J) \wedge \psi_2)\ . \end{array}$$

We transform all subformulae of the TCTL formula $\psi$ to be checked applying the above equivalences, resulting in the formula $\hat{\psi}$. Correctness of the transformation is straightforward for non-nested formulae. For nested formulae we need to slightly adapt the CTL model checking algorithm, as will be explained later (see Section 5.3.4).

**Example 5.4.** *The TCTL formula $E\mathcal{F}^{\leq 2}A\mathcal{G}^{[2,3]}a$ gets transformed into $E\mathcal{F}(z \leq 2 \wedge A\mathcal{G}(2 \leq z \leq 3 \to a)$.*

### 5.3.2 Finite State Space Abstraction

Since the state space of a timed automaton is in general infinite, to enable model checking we define a *finite abstraction* of the state space. In this abstraction we represent a (possibly infinite) number of states that behave "equivalent" by a single *abstract state*. That two states behave "equivalent" means, that no observation can distinguish between their behavior. Here we do not formalize the notion of *observation* and *observational equivalence*, neither the notion of *bisimulation*. Instead, we define that two states may (but do not have to) be equivalent only if they satisfy the same formulae of a given logic. This definition implies, that model checking the concrete system without abstraction would yield the same result as model checking the abstraction.

Up to the identity relation, an abstraction has in general less states than the concrete system. For this reason, abstraction is widely used also for finite-state systems, since model checking is faster and needs less memory for smaller systems than for larger ones. For infinite-state systems, for which state enumeration is not possible, abstraction may give us a finite-state system which can be model checked.

Before we deal with the abstraction for timed automata and TCTL, let us have a short look at abstractions for the simpler case of labeled state transition systems and the logic CTL$^*$. Assume a labeled state transition system $\mathcal{LSTS}$ with state set $\Sigma$, a set of atomic propositions $AP$, a labeling function $L : \Sigma \rightarrow 2^{AP}$, and two states $\sigma_1, \sigma_2 \in \Sigma$. The following conditions assure that $\sigma_1$ and $\sigma_2$ satisfy the same CTL$^*$ formulae:

- To satisfy the same atomic CTL$^*$ formulae, i.e., atomic propositions, $\sigma_1$ and $\sigma_2$ must be labeled with the same set of atomic propositions, i.e., $L(\sigma_1) = L(\sigma_2)$.
- To satisfy the same nested CTL$^*$ formulae, for each successor state of $\sigma_1$ there must be a successor state of $\sigma_2$ such that the two successor states again satisfy the same CTL$^*$ formulae, and vice versa, for each successor state of $\sigma_2$ there must be a successor state of $\sigma_1$ satisfying the same CTL$^*$ formulae. Thus we require that if there is a transition from $\sigma_1$ to a state $\sigma_1'$, than there is also a transition from $\sigma_2$ to a state $\sigma_2'$ that is equivalent to $\sigma_1'$, and vice versa.

Due to this inductive definition, we say that equivalent states can "mimic" each other's behavior in terms of atomic propositions.

The transition system $\mathcal{LSTS}$ may be parallel composed with other LSTSs. In this case label synchronization has to be considered. In order to be able to do the same synchronization steps from equivalent states, we extend the previous requirements as follows (the extensions are emphasized):

- As before, to satisfy the same atomic CTL$^*$ formulae, i.e., atomic propositions, $\sigma_1$ and $\sigma_2$ must be labeled with the same set of atomic propositions, i.e., $L(\sigma_1) = L(\sigma_2)$.
- We require that if there is a transition from $\sigma_1$ to a state $\sigma_1'$ *with label $a$*, than there is also a transition *with the same label $a$* from $\sigma_2$ to a state $\sigma_2'$ that is equivalent to $\sigma_1'$, and vice versa.

We say that equivalent states can "mimic" each other's behavior in terms of atomic propositions *and transition labels*. For a LSTS, a bisimulation is defined to be an equivalence relation on the state set satisfying the above conditions for each pair of equivalent states.

Let us try to extend the above conditions to timed automata and for the logic TCTL. Due to the discrete steps of timed automata, we will need similar conditions as above to cover atomic propositions and discrete steps. However, timed automata has additionally continuous steps, and TCTL may refer to atomic clock constraints. Thus we additionally require that equivalent states can mimic also the time steps of each other, and that equivalent states satisfy, in addition to atomic propositions, also the same atomic clock constraints.

Assume now a timed automaton with state space $\Sigma$. Two states $\sigma_1 = (\ell_1, \nu_1) \in \Sigma$ and $\sigma_2 = (\ell_2, \nu_2) \in \Sigma$ are equivalent, implying that they satisfy the same TCTL formulae, if the following conditions hold (the extensions are again emphasized):

- To satisfy the same atomic TCTL formulae, i.e., atomic propositions *and atomic clock constraints*, $\sigma_1$ and $\sigma_2$ must be labeled with the same set of atomic propositions, i.e., $L(\ell_1) = L(\ell_2)$, and *must satisfy the same atomic clock constraints.*
- We require that if there is a discrete transition from $\sigma_1$ to a state $\sigma_1'$ with label $a$, than there is also a discrete transition with label $a$ from $\sigma_2$ to a state $\sigma_2'$ that is equivalent to $\sigma_1'$, and vice versa.

- *For each time step from $\sigma_1$ in a successor state $\sigma_1'$ there is also a time step from $\sigma_2$ to some $\sigma_2'$ such that $\sigma_2'$ is equivalent to $\sigma_1'$, and vice versa.*

The above conditions are similar to the definition of *time-abstract bisimulation* (which does not consider atomic clock constraints). Note that for the time steps, the actual duration of the mimicking time step is not important, as long as the successor states cannot be distinguished by any TCTL formulae. This fact will become more clear later, when defining the abstraction for timed automata.

The above conditions would still lead to an infinite abstract state space, since there are infinitely many different clock constraints with different satisfying state sets. However, we need a *finite* abstraction to check *a certain TCTL property.* Consequently, equivalent states do not have to satisfy the same TCTL formulae but only the same subformulae of the given TCTL property. Thus we can release the requirements for *all* clock constraints to clock constraints appearing in the given timed automaton or in the given formula.

Assume a timed automaton $\mathcal{T}$ with locations $Loc$, clocks $\mathcal{C}$, and state space $\Sigma$. Assume furthermore an atomic proposition set $AP$, a labeling function $L : Loc \to 2^{AP}$, and a TCTL formula $\psi$. Below we define an abstraction by an equivalence relation $\cong \subseteq \Sigma \times \Sigma$ on the states of $\mathcal{T}$. We use

- $\lfloor r \rfloor$ to denote the integral part of $r \in \mathbb{R}$, i.e., $\max\{c \in \mathbb{N} \mid c \leq r\}$, and
- $fr(r)$ to denote the fractional part of $r \in \mathbb{R}$, i.e., $r - \lfloor r \rfloor$.

For clock constraints $x < c$ with $c \in \mathbb{N}$ we have:

$$\nu \models x < c \;\Leftrightarrow\; \nu(x) < c \;\Leftrightarrow\; \lfloor \nu(x) \rfloor < c.$$

For clock constraints $x \leq c$ with $c \in \mathbb{N}$ we have:

$$\nu \models x \leq c \;\Leftrightarrow\; \nu(x) \leq c \;\Leftrightarrow\; \lfloor \nu(x) \rfloor < c \vee (\lfloor \nu(x) \rfloor = c \wedge fr(\nu(x)) = 0)\,.$$

That means, if we would require that equivalent states should satisfy the same clock constraints over the clock set $\mathcal{C}$, then only states $(\ell, \nu)$ and $(\ell, \nu')$ satisfying

$$\lfloor \nu(x) \rfloor = \lfloor \nu'(x) \rfloor \textit{ and } fr(\nu(x)) = 0 \textit{ iff } fr(\nu'(x)) = 0$$

for all $x \in \mathcal{C}$ could be equivalent. However, as mentioned above, if we distinguish between all possible integral parts in $\mathbb{N}$, we would generate infinitely many equivalence classes.

Given the timed automaton $\mathcal{T}$ and the TCTL formula $\psi$, we are only interested in those clock constraints that play a role in the satisfaction or violation of $\psi$ by $\mathcal{T}$. I.e., it is sufficient if equivalent states satisfy the same clock constraints occurring in $\mathcal{T}$ or $\psi$.

Let $c_x$ be the largest constant which a clock $x$ is compared to in $\mathcal{T}$ or in $\psi$. Then there is no observation which could distinguish between the $x$-values in $(\ell, \nu)$ and $(\ell, \nu')$ if $\nu(x) > c_x$ and $\nu'(x) > c_x$. I.e., equivalent states $(\ell, \nu) \cong (\ell, \nu')$ should satisfy

$$\begin{aligned} &(\nu(x) > c_x \wedge \nu'(x) > c_x) \quad \vee \\ &(\lfloor \nu(x) \rfloor = \lfloor \nu'(x) \rfloor \;\wedge\; fr(\nu(x)) = 0 \textit{ iff } fr(\nu'(x)) = 0) \end{aligned} \tag{5.1}$$

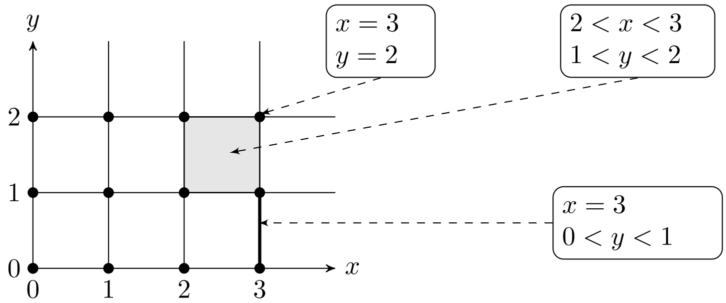
for all $x \in \mathcal{C}$.

**Example 5.5.** *Assume that $\mathcal{T}$ has two clocks $x$ and $y$ with $c_x = 3$ and $c_y = 2$, i.e., the largest constant that $x$ is compared to in $\mathcal{T}$ or in $\psi$ is 3, and for $y$ this is 2.*

*Then we can possibly observe different behavior for states satisfying $x = 0$, $0 < x < 1$, $x = 1$, $1 < x < 2$, $x = 2$, $2 < x < 3$, $x = 3$, and $x > 3$. I.e., two states that satisfy two different clock constraints from the above list must not be equivalent.*

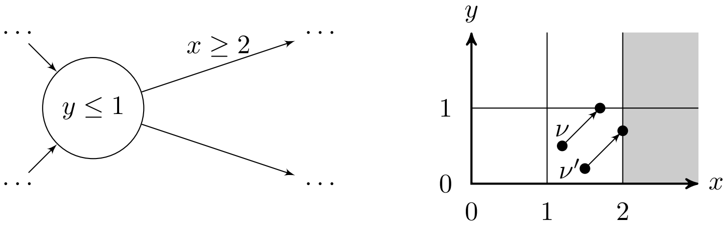
*Similarly for $y$, only states satisfying the same clock constraint from the list $y = 0$, $0 < y < 1$, $y = 1$, $1 < y < 2$, $y = 2$, and $y > 2$ may be equivalent.*

*In the graphical representation below, valuations belonging to different points, line fragments, or boxes must not be equivalent. This yields at least 48 equivalence classes.*



As the following example illustrates, we must make a further refinement of the abstraction.

**Example 5.6.** *Assume the following fraction of a timed automaton and the corresponding classification of states according to the above observations:*



*If the control is in location $l$ with a valuation $\nu$ with, e.g., $\nu(x) = 1.2$ and $\nu(y) = 0.5$, then the transition with condition $x \geq 2$ cannot be taken, since the invariant $y \leq 1$ forces the control to leave the location before the value of $x$ reaches 2. But if the valuation assigns, e.g., $\nu'(x) = 1.5$ and $\nu'(y) = 0.2$, then the transition gets enabled before the invariant gets violated.*

*Though the classification respects Equation 5.1, the valuations in the classes are not yet of the same behavior.*

What we need is a refinement taking the order of the fractional parts of the clock values into account. I.e., we must extend the condition of Equation 5.1 with the requirement that states $(\ell, \nu)$ and $(\ell, \nu')$ may be equivalent only if for all clock pairs $x, y \in \mathcal{C}$ with $\nu(x), \nu'(x) \leq c_x \wedge \nu(y), \nu'(y) \leq c_y$

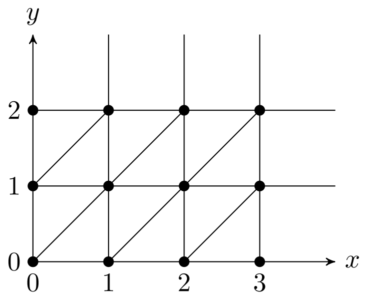
$$\begin{array}{lcll} \mathit{fr}(\nu(x)) < \mathit{fr}(\nu(y)) & \mathit{iff} & \mathit{fr}(\nu'(x)) < \mathit{fr}(\nu'(y)) & \wedge \\ \mathit{fr}(\nu(x)) = \mathit{fr}(\nu(y)) & \mathit{iff} & \mathit{fr}(\nu'(x)) = \mathit{fr}(\nu'(y)) & \wedge \\ \mathit{fr}(\nu(x)) > \mathit{fr}(\nu(y)) & \mathit{iff} & \mathit{fr}(\nu'(x)) > \mathit{fr}(\nu'(y)). & \end{array}$$

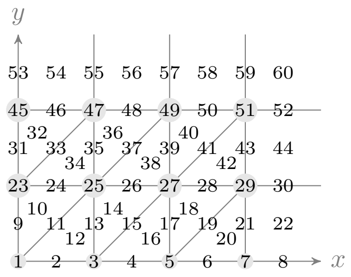
Because of symmetry requiring

$$\mathit{fr}(\nu(x)) \leq \mathit{fr}(\nu(y)) \quad \mathit{iff} \quad \mathit{fr}(\nu'(x)) \leq \mathit{fr}(\nu'(y)). \tag{5.2}$$

is sufficient.

**Example 5.7.** *We extend the graphical representation of the clock equivalence classes from Example 5.5 taking the conditions of both Equations 5.1 and 5.2 into account. Below, the left picture shows the division of the state space into regions, whereas the right picture enumerates the resulting regions.*





$\cong$ **Definition 5.12.** *For a timed automaton $\mathcal{T}$ and a TCTL formula $\psi$, both over a clock set $\mathcal{C}$, we define the* clock equivalence *relation $\cong \subseteq \Sigma \times \Sigma$ by $(\ell, \nu) \cong (\ell', \nu')$ iff $\ell = \ell'$ and*

- *for all $x \in \mathcal{C}$, either $\nu(x) > c_x \wedge \nu'(x) > c_x$ or*

$$\lfloor \nu(x) \rfloor = \lfloor \nu'(x) \rfloor \ \wedge \ (\mathit{fr}(\nu(x)) = 0 \quad \mathit{iff} \quad \mathit{fr}(\nu'(x)) = 0)$$

- *for all $x, y \in \mathcal{C}$ if $\nu(x), \nu'(x) \leq c_x$ and $\nu(y), \nu'(y) \leq c_y$ then*

$$\mathit{fr}(\nu(x)) \leq \mathit{fr}(\nu(y)) \quad \mathit{iff} \quad \mathit{fr}(\nu'(x)) \leq \mathit{fr}(\nu'(y)).$$

*The* clock region *of an evaluation $\nu \in V$ is the set $[\nu] = \{\nu' \in V \mid \nu \cong \nu'\}$. The* state region *of a state $(\ell, \nu) \in \Sigma$ is the set $[(\ell, \nu)] = \{(\ell, \nu') \in \Sigma \mid \nu \cong \nu'\}$. We also write $(\ell, r)$ for $\{(\ell, \nu) \mid nu \in r\}$.*

### 5.3.3 The Region Transition System

After we have defined state regions, next we define how to connect them by abstract transitions, yielding an abstract transition system, which we call the region transition system.

We extend the satisfaction relation for clock constraints to regions by defining

$$\begin{aligned} r \models g \quad & \mathit{iff} \quad \forall \nu \in r.\ \nu \models g \\ (\ell, r) \models g \quad & \mathit{iff} \quad r \models g. \end{aligned}$$

for $r$ being a clock region of $\mathcal{T}$ with clocks $\mathcal{C}$ and a TCTL formula $\psi$, and $g \in ACC_{\mathcal{C}} \cup ACC_{\psi}$. On the right-hand side, instead of the universal quantification we could have also required just the existence of a valuation in $r$ satisfying $g$, as it holds that

$$\forall \nu, \nu' \in r.\ \nu \models g \ \leftrightarrow \ \nu' \models g.$$

We also extend the reset operator to regions as follows:

$$\mathit{reset}(C) \ \mathit{in}\ r \quad = \quad \{(\ell, \mathit{reset}(C) \ \mathit{in}\ \nu) \in \Sigma \mid (\ell, \nu) \in r\}.$$

Note that $\mathit{reset}(C)\ \mathit{in}\ r$ is again a region.

**Definition 5.13.** *The clock region* $r_\infty = \{\nu \in V \mid \forall x \in \mathcal{C}.\ \nu(x) > c_x\}$ *is called* unbounded. *Let* $r, r'$ *be two clock regions. The region* $r'$ *is the* successor clock region *of* $r$*, denoted by* $r' = succ(r)$*, if either*

- $r = r' = r_\infty$*, or*
- $r \neq r_\infty$*,* $r \neq r'$*, and for all* $\nu \in r$*:*

$$\exists d \in \mathbb{R}_{>0}.\ (\nu + d \in r' \ \wedge\ \forall 0 \leq d' \leq d.\ \nu + d' \in r \cup r').$$

*The* successor state region *is defined as* $succ((\ell, r)) = (\ell, succ(r))$*.*

**Definition 5.14.** *Let* $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$ *be a non-Zeno timed automaton and let* $\hat{\psi}$ *be an unbounded TCTL formula over* $\mathcal{C}$ *and a set* $AP$ *of atomic propositions. The region transition system of* $\mathcal{T}$ *for* $\hat{\psi}$ *is a labeled state transition system* $\mathcal{RTS}(\mathcal{T}, \hat{\psi}) = (\Sigma', Lab', Edge', Init')$ *with*

- $\Sigma'$ *the finite set of all state regions,*
- $Lab' = Lab \cup \{\tau\}$*,*
- $Init' = \{[\sigma] \mid \sigma \in Init\}$*,*

*and*

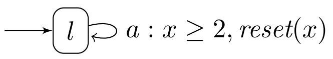
$$\frac{\begin{array}{c}(\ell, a, (g, \mathcal{C}), \ell') \in Edge \\ r \models g \quad r' = reset(\mathcal{C})\ in\ r \quad r' \models Inv(\ell')\end{array}}{(\ell, [\nu]) \xrightarrow{a} (\ell', [\nu'])} \quad \texttt{Rule}_{\ \textit{discrete}}$$

$$\frac{r \models Inv(\ell) \quad succ(r) \models Inv(\ell)}{(\ell, r) \xrightarrow{\tau} (\ell, succ(r))} \quad \texttt{Rule}_{\ \textit{time}}$$

*Assume a labeling function* $L : \Sigma \to 2^{AP}$ *of* $\mathcal{T}$*. We define*

- $AP' = AP \cup ACC(\mathcal{T}) \cup ACC(\psi)$
- $L'((\ell, r)) = L(\ell) \cup \{g \in AP' \backslash AP \mid r \models g\}$

**Example 5.8.** *Assume the following timed automaton having a single clock* $x$*:*



*Without taking any TCTL formula into account, the abstraction distinguishes the following equivalence classes:*

$$\begin{array}{rcl}
r_{[0,0]} & = & \{(\ell, \nu) \in \Sigma \mid \nu(x) = 0\} \\
r_{(0,1)} & = & \{(\ell, \nu) \in \Sigma \mid 0 < \nu(x) < 1\} \\
r_{[1,1]} & = & \{(\ell, \nu) \in \Sigma \mid \nu(x) = 1\} \\
r_{(1,2)} & = & \{(\ell, \nu) \in \Sigma \mid 1 < \nu(x) < 2\} \\
r_{[2,2]} & = & \{(\ell, \nu) \in \Sigma \mid \nu(x) = 2\} \\
r_{(2,\infty)} & = & \{(\ell, \nu) \in \Sigma \mid \nu(x) > 2\}
\end{array}$$

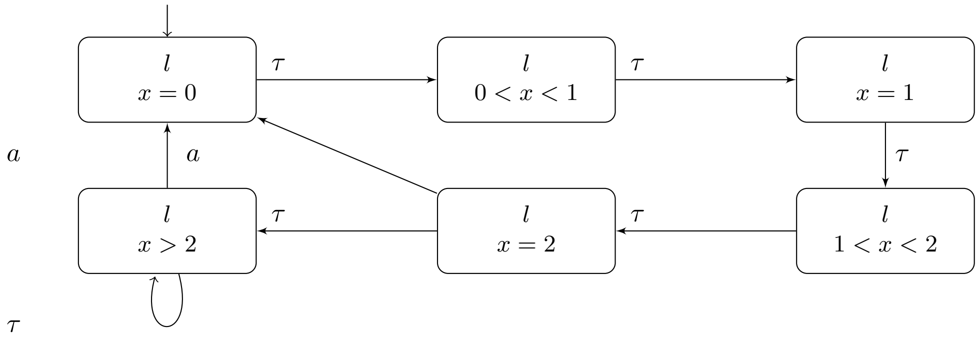
*For the transitions,* $\tau$*-transitions are defined from each region into its successor region:*

$$\begin{array}{lll}
r_{[0,0]} \xrightarrow{\tau} r_{(0,1)} & r_{(0,1)} \xrightarrow{\tau} r_{[1,1]} & r_{[1,1]} \xrightarrow{\tau} r_{(1,2)} \\
r_{(1,2)} \xrightarrow{\tau} r_{[2,2]} & r_{[2,2]} \xrightarrow{\tau} r_{(2,\infty)} & r_{(2,\infty)} \xrightarrow{\tau} r_{(2,\infty)}
\end{array}$$

*Discrete transitions are possible from the regions with $x \geq 2$ into the region with $x = 0$:*

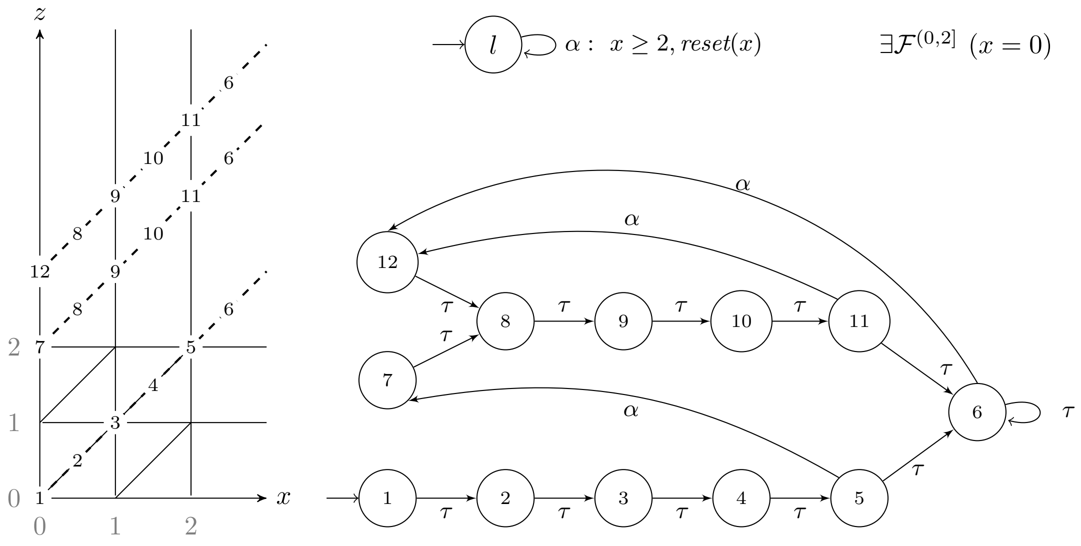
$$r_{[2,2]} \xrightarrow{a} r_{[0,0]} \qquad r_{(2,\infty)} \xrightarrow{a} r_{[0,0]}$$

*The resulting region transition graph can be visualized as follows, where for clarity we write into the states the locations and the constraints to which they correspond:*



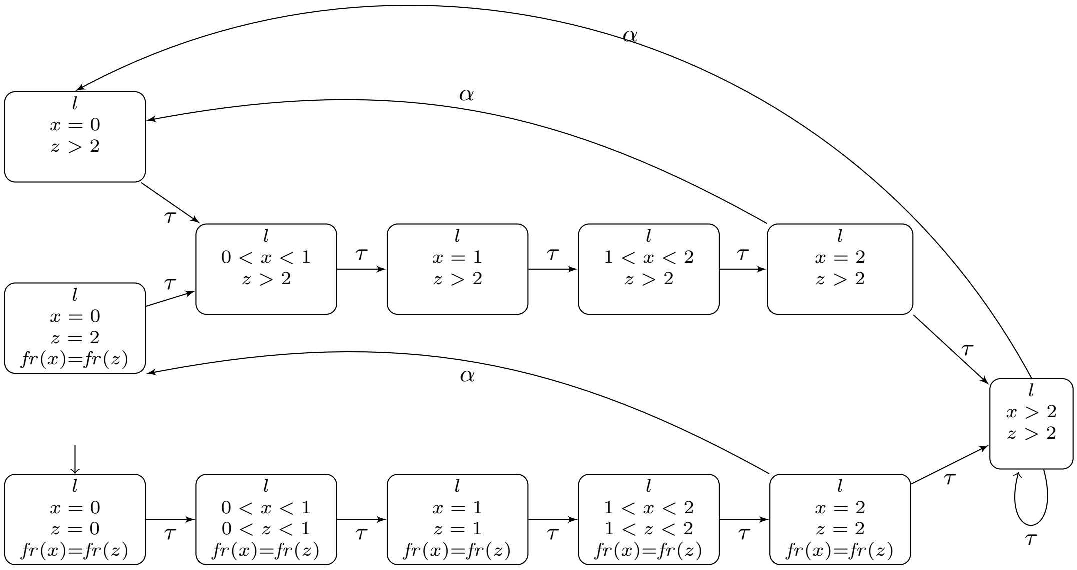
**Example 5.9.** *Assume the same timed automaton as in the previous Example 5.8, but now additionally consider the TCTL formula $E\mathcal{F}^{(0,2]}(x = 0)$. After removing the bound we get the unbounded formula $E\mathcal{F}(0 < z \leq 2 \wedge x = 0)$. Thus we have $c_x = 2$ and $c_z = 2$.*

*We get the following region transition system, where we omit unreachable abstract states. Dotted lines in the coordinate system represent possible behaviors, moving through the different regions.*



$z$
$x$
0
1
2
0
1
2
1
2
3
4
5
6
7
8
9
10
11
12
$l$
$\alpha : \ x \geq 2, reset(x)$
$\exists\mathcal{F}^{(0,2]} \ (x = 0)$
$\alpha$
$\alpha$
$\alpha$
$\tau$

*The following graph shows again the region transition system where the abstract states are annotated with the information determining the regions:*



α
α
α
l, x = 0, z > 2
l, 0 < x < 1, z > 2
l, x = 1, z > 2
l, 1 < x < 2, z > 2
l, x = 2, z > 2
l, x = 0, z = 2, fr(x)=fr(z)
l, x > 2, z > 2
l, x = 0, z = 0, fr(x)=fr(z)
l, 0 < x < 1, 0 < z < 1, fr(x)=fr(z)
l, x = 1, z = 1, fr(x)=fr(z)
l, 1 < x < 2, 1 < z < 2, fr(x)=fr(z)
l, x = 2, z = 2, fr(x)=fr(z)
τ

The next lemma states that infinite time-convergent paths of a timed automaton correspond to finite paths in the region transition system.

**Lemma 5.1.** *For non-Zeno $\mathcal{T}$ and $\pi = s_0 \to s_1 \to \dots$ an infinite path of $\mathcal{T}$:*

- *if $\pi$ is time-convergent, then there is an index $j$ and a state region $(\ell, r)$ such that $s_i \in (\ell, r)$ for all $i \geq j$.*
- *if there is a state region $(\ell, r)$ with $r \neq r_\infty$ and an index $j$ such that $s_i \in (\ell, r)$ for all $i \geq j$ then $\pi$ is time-convergent.*

**Theorem 5.2.** *A non-Zeno timed automaton $\mathcal{T}$ is timelock free iff its region transition system does not have any deadlocks, i.e., reachable terminal states.*

### 5.3.4 TCTL Model Checking

The procedure is quite similar to CTL model checking for finite automata. The only difference concerns the handling of nested time bounds in TCTL formulae.

As in CTL model checking, we label the abstract states of the region transition system with subformulae of the formula $\psi$ to be checked, inside-out starting with the inner-most subformulae. However, since we want to use a single auxiliary clock, we must additionally represent the "restart" of the auxiliary clock at some places.

To explain the problem, consider the formula $E\mathcal{F}^{[0,1]}(a \wedge E\mathcal{F}^{[1,2]}b)$. Removing the bounds yields $E\mathcal{F}(0 \leq z \leq 1 \wedge a \wedge E\mathcal{F}(1 \leq z \leq 2 \wedge b))$. The labeling with the atomic propositions $a$ and $b$ is defined by the labeling function. The labeling with atomic clock constraints is done upon the generation of the region transition system. The first step of the model checking algorithm would label those regions with $1 \leq z \leq 2 \wedge b$ that are labeled with $1 \leq z$, $z \leq 2$, and $b$. Now we come to the more interesting part: the algorithm would determine all those regions from which a region labeled with $1 \leq z \leq 2 \wedge b$ is reachable, and may label them with $E\mathcal{F}(1 \leq z \leq 2 \wedge b)$. Now we make two observations: Firstly, $E\mathcal{F}^{[1,2]}b$ is satisfied only by those determined regions that are labeled with $z = 0$. Secondly, the start value $z = 0$ of the auxiliary clock is just a convention, we could also have started with a value, e.g., $z = 2$ and check reachability of $3 \leq z \leq 4 \wedge b$. Consequently, we should label all those regions $r$ with $E\mathcal{F}^{[1,2]}b$ for that the region $\mathit{reset}(z)\ \mathit{in}\ r$ is labeled with $E\mathcal{F}(1 \leq z \leq 2 \wedge b)$. The labeling for the other subformulae is analogous. After termination, the timed automaton satisfies the above TCTL formula iff each initial region is labeled with it.

**Lemma 5.2.** *For a non-Zeno timed automaton $\mathcal{T}$ and an unbounded TCTL formula $\psi$:*

$$\mathcal{T} \models_{TCTL} \psi \quad \textit{iff} \quad \mathcal{RTS}(\mathcal{T}, \psi) \models_{CTL} \hat{\psi}$$

**Lemma 5.3.** *The model checking problem for timed automata and TCTL properties is complete for PSPACE.*

# Exercises

**Exercise 5.1.** Construct the automaton $\mathcal{H}_{\text{Controller}}||\mathcal{H}_{\text{Gate}}$ from Example 5.3.

**Exercise 5.2.** Prove Theorem 5.2.

# Chapter 6

# Rectangular Automata

In the previous chapter we have seen that TCTL for timed automata, a special class of hybrid automata, is decidable, thus model checking is possible. In this chapter we discuss a bit more general class, the class of *rectangular automata*, and analyse decidability. The contents of this chapter are based on [HKPV98].

Rectangular automata build an interesting class of hybrid automata because on the one hand they allow a more expressive modeling than timed automata and on the other hand (under some additional conditions) both safety and liveness for rectangular automata are decidable. However, they lie on the boundary of decidability in the sense that several slight generalizations lead to undecidability.

In the previous chapters we used temporal logics supporting the specification of both safety and liveness properties. From now on we restrict ourselves to *safety* properties, stating that each reachable state of an automaton is included in a given set of safe states.

In the following Section 6.1 we define syntax and semantics of rectangular automata, before discussing decidability in Section 6.2.

## 6.1 Syntax and Semantics

In the following we first formally define the syntax and semantics of rectangular automata. As rectangular automata are special hybrid automata, their states $\sigma = (l, \nu) \in \Sigma = Loc \times V$ also consist of a discrete component describing the current location, and of a valuation component, assigning values to the real-valued variables. To simplify the notation, in the following we assume that the real-valued variables $Var = \{x_1, \ldots, x_d\}$ of the automata are ordered and write $(l, v) \in Loc \times \mathbb{R}^d$ for a state $(l, \nu)$ with $\nu(x_i) = v_i$ for all $i = 1, \ldots, d$.

To define rectangular automata we first need to define rectangular sets.

**Definition 6.1 (Rectangular set).** *A set $\mathcal{R} \subset \mathbb{R}^d$ is* rectangular *if it is a cartesian product of (possibly unbounded) intervals, all of whose finite endpoints are rational. The set of rectangular sets in $\mathbb{R}^d$ is denoted by $\mathcal{R}^d$.*
*Given a set Loc of locations, a subset of the state space $Loc \times \mathbb{R}^d$ is called a* zone*. Each zone $Z$ is decomposable into a collection $\bigcup_{l \in Loc} \{l\} \times Z_l$ of zones. The zone $Z$ is* rectangular *iff each $Z_l$ is rectangular. A zone is* multirectangular*, if it is a finite union of rectangular zones.*

Rectangular automata are hybrid automata whose invariants, activities, and transition relations are all described by rectangular sets. For the invariants and transition guards it means that those conditions may not compare the values of different variables to each other, but to constant values only. Similarly, a transition may reset the value of a variable to a non-deterministically chosen value from an interval, whose end-points are constants, i.e., they do not depend on the values of other variables. Finally, the activities assign constant lower and upper bounds to the derivatives, allowing also non-linear behaviour. However, since the evolution of a variable may not depend on the value of another variable, the set of states reachable via time steps from a rectangular set is again a rectangular set.

**Definition 6.2 (Syntax of rectangular automata).** *A $d$-dimensional* rectangular automaton *(or short rectangular automaton) is a tuple $\mathcal{H} = (Loc, Var, Con, Lab, Edge, Act, Inv, Init)$ with*

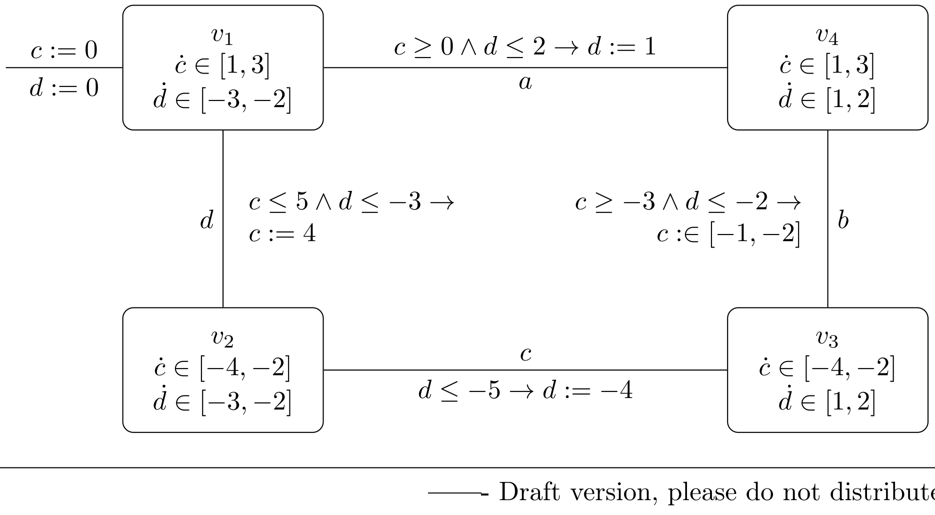
- *a finite set $Loc$ of locations;*
- *a finite set $Var = \{x_1, \ldots, x_d\}$ of $d$ ordered real-valued variables; we write $x = (x_1, \ldots, x_d)$ for the ordered sequence of the variables;*
- *a function $Con : Loc \to 2^{Var}$ assigning a set of controlled variables to each location;*
- *a finite set $Lab$ of synchronization labels;*
- *a set $Edge \subseteq Loc \times Lab \times (\mathcal{R}^d \times \mathcal{R}^d \times 2^{\{1,\ldots,n\}}) \times Loc$ of edges;*
- *a flow function $Act : Loc \to \mathcal{R}^d$;*
- *an invariant function $Inv : Loc \to \mathcal{R}^d$;*
- *initial states $Init : Loc \to \mathcal{R}^d$.*

*A rectangular automaton is* initialized *iff for all edges $e = (l, a, pre, post, jump, l') \in Edge$ and all $i \in \{1, \ldots, n\}$ we have that $Act(l)_i \neq Act(l')_i$ implies $i \in jump$, where $Act(l)_i$ is the projection of $Act(l)$ to the $i$th dimension.*

For the flows, the first time derivatives of the flow trajectories in location $l \in Loc$ are within the rectangular set $Act(l)$. For the jumps, an edge $e = (l, a, pre, post, jump, l') \in Edge$ may move control from location $l$ to location $l'$ starting from a valuation in $pre$, changing the value of each variable $x_i \in jump$ to a nondeterministically chosen value from $post_i$ (the projection of $post$ to the $i$th dimension), and leaving the values of the other variables unchanged.

An initialized rectangular automaton has the property that whenever the flow of a variable changes due to a discrete transition, the variable is re-initialized to a value from an interval with constant bounds. The reachability problem for initialized rectangular automata is decidable. However, it becomes undecidable if the restriction of being initialized is relaxed.

**Example 6.1.** *The following graph illustrates is an initialized rectangular automaton:*



Note that a timed automaton is a special rectangular automaton such that every variable is a clock, the initial sets $Init(l)$ are empty or are singletons for each location $l \in Loc$, and the edges reset variables to 0 only. Furthermore, if we replace rectangular regions with linear regions, we obtain linear hybrid automata, a super-class of rectangular automata, which are the subject of the next chapter.

The semantics of rectangular automata is derived from the semantics of hybrid automata as follows.

**Definition 6.3 (Semantics of rectangular automata).** *The* operational semantics *of a rectangular automaton $\mathcal{H} = (Loc, Var, Con, Lab, Edge, Act, Inv, Init)$ is given by the following two rules:*

$$\frac{\begin{array}{c}(l, a, pre, post, jump, l') \in Edge \\ v \in pre \quad v' \in post \quad \forall i \notin jump. v'_i = v_i \quad v' \in Inv(l')\end{array}}{(l, v) \xrightarrow{a} (l', v')} \quad \texttt{Rule}_{\;\texttt{discrete}}$$

$$\frac{(t = 0 \wedge v = v') \vee (t > 0 \wedge (v' - v)/t \in Act(l)) \quad v' \in Inv(l)}{(l, v) \xrightarrow{t} (l, v')} \quad \texttt{Rule}_{\;\texttt{time}}$$

*The* one-step transition *is given by $\rightarrow = \xrightarrow{a} \cup \xrightarrow{t}$, its* transitive closure *by $\rightarrow^*$. A* path *is a sequence $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ starting in an initial state $\sigma_0 = (\ell_0, v_0)$ with $v_0 \in Init(\ell_0) \cap Inv(\ell_0)$. A state is* reachable *iff there exists a path leading to it.*

Note that, similarly to timed automata, the invariant sets of rectangular automata are convex. Furthermore, though the time behaviour can be non-linear, for each non-linear time flow there is a corresponding linear one leading to the same state in the same time. Thus for the time steps we do not need to require the invariant to hold at each time point during the time step, but it is sufficient to require that the invariant holds initially and after each step.

**Lemma 6.1.** *For every multirectangular zone $Z$ of a d-dimensional rectangular automaton $\mathcal{H}$, and every label $lab \in Lab \cup \mathbb{R}_{\geq 0}$, the zones $Post^{lab}(Z) = \{(l', v') \in Loc \times \mathbb{R}^d \mid \exists (l, v) \in Z.\ (l, v) \xrightarrow{lab} (l', v')\}$ and $Pre^{lab}(Z) = \{(l, v) \in Loc \times \mathbb{R}^d \mid \exists (l', v') \in Z.\ (l, v) \xrightarrow{lab} (l', v')\}$ are multirectangular.*

*Proof.* It suffices to prove the lemma for elementary regions of the form $Z = (\{l\}, \mathcal{R})$ with $\mathcal{R}$ rectangular. We distinguish between discrete and time steps.

For discrete steps assume $lab = a \in Lab$. Let furthermore $e = (l, a, pre, post, jump, l')$ be an edge. Then $Post^a(Z) = \{l'\} \times S$ with

$$S_i = \begin{cases} \mathcal{R}_i \cap pre_i \cap post_i \cap Inv(l')_i & \textit{if } i \notin jump, \\ post_i \cap Inv(l')_i & \textit{if } i \in jump \textit{ and } \mathcal{R}_i \cap pre_i \neq 0, \\ \emptyset & \textit{if } i \in jump \textit{ and } \mathcal{R}_i \cap pre_i = 0. \end{cases}$$

Thus $Post^a(Z)$ is rectangular, and the union over all edges starting in $l$ with label $a$ is a multirectangular zone.

For time steps, if $lab = 0$ then $Post^0(Z) = Z$. Thus assume $lab = t \in \mathbb{R}$ with $t > 0$. Let $L = inf(\mathcal{R}_i) + t \cdot inf(Act(l)_i)$ and $U = sup(\mathcal{R}_i) + t \cdot sup(Act(l)_i)$.

Then $Post^t(Z) = \{l\} \times S$ with

$$S_i = \begin{cases} Inv(l)_i \cap [L, \infty) \cap (-\infty, U] & \textit{if } \mathcal{R}_i \textit{ and } Act(l)_i \textit{ are closed,} \\ Inv(l)_i \cap (L, \infty) \cap (-\infty, U] & \textit{if } \mathcal{R}_i \textit{ or } Act(l)_i \textit{are left-open and} \\ & \textit{both are right-closed, and} \\ Inv(l)_i \cap [L, \infty) \cap (-\infty, U) & \textit{if } \mathcal{R}_i \textit{ or } Act(l)_i \textit{are right-open and} \\ & \textit{both are left-closed.} \end{cases}$$

Thus $Post^t(Z)$ is a rectangular zone. □ □

Note that the reachable zone of a rectangular automaton is in general an infinite union of rectangular zones, and may thus be not multirectangular.

## 6.2 Decidability of Rectangular Automata

The reachability problem for initialized rectangular automata is decidable.

**Lemma 6.2.** *The reachability problem for initialized rectangular automata is PSPACE complete.*

The proof makes use of the fact that the reachability problem for timed automata is complete for PSPACE. It defines a transformation of initialized rectangular automata to timed automata thereby proving PSPACE completeness. The transformation is done in three steps:

Timed automaton
↑
Initialized stopwatch automaton
↑
Initialized singular automaton
↑
Initialized rectangular automaton

In the following we describe these steps. Note that the transformation does not only prove decidability, but also gives us a model checking algorithm for initialized rectangular automata, since we can apply the previously discussed model checking algorithm to the resulting timed automaton.

### 6.2.1 From Initialized Stopwatch Automata to Timed Automata

Let us start with the first step transforming an initialized stopwatch automaton into a timed automaton.

**Definition 6.4.**
- *A rectangular automaton has* deterministic jumps*, if (1) $Init(l)$ is empty or a singleton for all $l$, and (2) the post-interval for each variable from the jump-set of each edge is a singleton.*
- *A* stopwatch *is a variable with derivatives* 0 *or* 1 *only.*
- *A* stopwatch automaton *is a rectangular automaton with deterministic jumps and stopwatch variables only.*

Initialized stopwatch automata can be polynomially encoded by timed automata, as shown below. This implies the decidability of initialized stopwatch automata. However, the reachability problem for non-initialized stopwatch automata is undecidable.

**Lemma 6.3.** *The reachability problem for initialized stopwatch automata is PSPACE complete.*

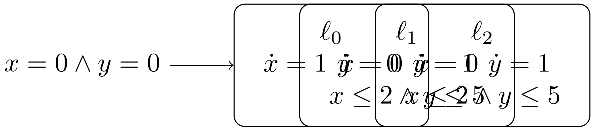
The encoding works as follows. First notice, that a timed automaton is a stopwatch automaton such that every variable is a clock.

Assume that $\mathcal{H}$ is a $d$-dimensional initialized stopwatch automaton with location set $Loc_{\mathcal{H}}$. Let $\kappa$ be the set of rational constants used in the definition of $\mathcal{H}$, and let $\kappa_{\perp} = \kappa \cup \{\perp\}$.

We define a $d$-dimensional timed automaton $\mathcal{H}'$ with locations $Loc_{\mathcal{H}'} = Loc_{\mathcal{H}} \times \kappa_{\perp}^{\{1,\ldots,d\}}$. Each location $(l, f)$ of $\mathcal{H}'$ consists of a location $l$ of $\mathcal{H}$ and a function $f : \{1, \ldots, d\} \to \kappa_{\perp}$. Each state $q = ((l, f), v)$ of $\mathcal{H}'$ represents the state $\alpha(q) = (l, u)$ of $\mathcal{H}$, where $u_i = v_i$ if $f(i) = \perp$, and $u_i = f(i)$ if $f(i) \neq \perp$.

Intuitively, if the $i$th stopwatch of $\mathcal{H}$ is running (slope 1), then its value is tracked by the value of the $i$th clock of $\mathcal{H}'$; if the $i$th stopwatch is halted (slope 0) at value $k \in \kappa$, then this value is remembered by the current location of $\mathcal{H}'$.

**Example 6.2.** *Consider the following initialized stopwatch automaton:*



*This automaton can be transformed to the following timed automaton:*

### 6.2.2 From Initialized Singular Automata to Initialized Stopwatch Automata

**Definition 6.5.**

- *A variable $x_i$ is a* finite-slope variable *if flow$(l)_i$ is a singleton in all locations $l$.*
- *A* singular automaton *is a rectangular automaton with deterministic jumps such that every variable of the automaton is a finite-slope variable.*

**Lemma 6.4.** *The reachability problem for initialized singular automata is PSPACE complete.*

The proof is again based on automata transformation. Initialized singular automata can be rescaled to initialized stopwatch automata as follows.

Let $B$ be a $d$-dimensional initialized singular automaton with $\epsilon$-moves. We define a $d$-dimensional initialized stopwatch automaton $C_B$ with the same location set, edge set, and label set as $B$.

Each state $q = (l, v)$ of $C_B$ corresponds to the state $\beta(q) = (l, \beta(v))$ of $B$ with $\beta : \mathbb{R}^d \to \mathbb{R}^d$ defined as follows:

For each location $l$ of $B$, if $Act_B(l) = \Pi_{i=1}^{d}[k_i, k_i]$, then $\beta(v_1, \ldots, v_d) = (\ell_1 \cdot v_1, \ldots, \ell_d \cdot v_d)$ with $\ell_i = k_i$ if $k_i \neq 0$, and $\ell_i = 1$ if $k_i = 0$;

$\beta$ can be viewed as a rescaling of the state space. All conditions in the automaton $B$ occur accordingly rescaled in $C_B$.

The reachable set $Reach(B)$ of $B$ is $\beta(Reach(C_B))$.

### 6.2.3 From Initialized Rectangular Automaton to Initialized Singular Automaton

**Lemma 6.5.** *The reachability problem for initialized rectangular automata is PSPACE complete.*

The proof is based on the translation of a $d$-dimensional initialized rectangular automaton $\mathcal{H}$ into a $(2n+1)$-dimensional initialized singular automaton $B$, such that $B$ contains all reachability information about $\mathcal{H}$.

The translation is similar to the subset construction for determinizing finite automata.

The idea is to replace each variable $c$ of $\mathcal{H}$ by two finite-slope variables $c_l$ and $c_u$: $c_l$ tracks the least possible value of $c$, and $c_u$ tracks the greatest possible value of $c$.

## Exercises

**Exercise 6.1.** A gas burner is a device to generate a flame to heat up products using a gaseous fuel. We assume there is a gas burner, such that

(a) any leakage of it can be detected and stopped within **1** second, and

(b) it will not leak for at least **30** seconds after a leakage has been stopped or *from beginning*.

The gas burner also records the cumulative *leakage time* and the *total elapsed time*.

(1) Please model the gas burner by a rectangular automaton and try to keep it as simple as possible.

(2) Is it possible to have a 2-second leakage time in 70 seconds? If so, please give a sample execution.

# Chapter 7

# Linear Hybrid Automata

In this chapter we discuss a further class of hybrid automata called *linear hybrid automata*. Linear hybrid automata are time-deterministic hybrid automata whose definitions contain linear terms, only. They are more expressive than timed or rectangular automata, and the reachability problem for linear hybrid automata is in general undecidable. However, *bounded* reachability, i.e., reachability within a fixed number of steps, is still decidable and can be efficiently computed. Approximation and minimization techniques can be additionally used for the successful analysis of linear hybrid automata.

We introduce linear hybrid automata in Section 7.1. Forward and backward analysis techniques are discussed in the Sections 7.2 and 7.3, respectively. Approximation methods for linear hybrid automata are described in Section 7.4, and we handle minimization in Section 7.5

The contents of this chapter are based on [ACH$^+$95].

## 7.1 Syntax and Semantics

**Definition 7.1.**
- *A* linear term *over the set Var of variables is a linear combination of variables in Var with integer (rational) coefficients.*
- *A* linear formula *over Var is a Boolean combination of (in)equalities between linear terms over Var.*
- *A hybrid automaton is* time deterministic *iff for every location $l \in Loc$ and every valuation $\nu \in V$ there is at most one activity $f \in Act(l)$ with $f(0) = \nu$. The activity $f$, then, is denoted by $f_l[\nu]$, its component for $x \in Var$ by $f_l^x[\nu]$.* $f_l[\nu]$ $f_l^x[\nu]$

The restrictions on the syntax of linear hybrid automata affect the activities, the invariants, and the discrete edges.

**Definition 7.2 (Syntax of linear hybrid automata).** *A* linear hybrid automaton *is a time-deterministic hybrid automaton with the following properties:*

- Activities $Act(l)$ *are given as sets of differential equations $\dot{x} = k_x$, one for each variable $x \in Var$, with $k_x$ an integer (rational) constant:*

$$f_l^x[\nu](t) = \nu(x) + k_x \cdot t.$$

- Invariants $Inv(l)$ *are defined by linear formulae $\psi$ over Var:*

$$\nu \in Inv(l) \quad \textit{iff} \quad \nu \models \psi$$

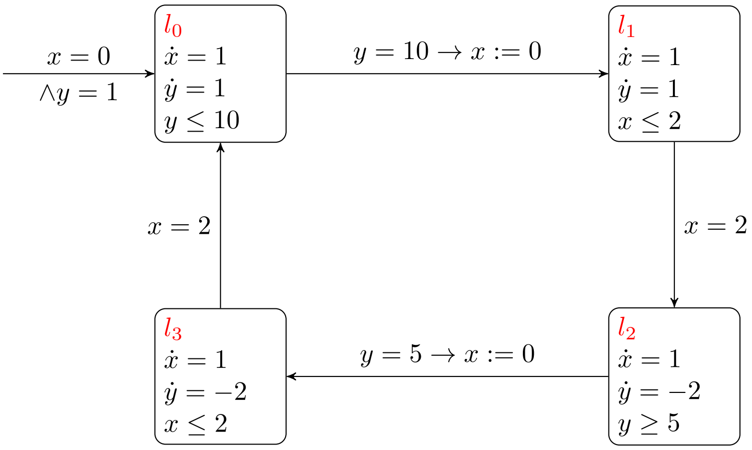


Figure 7.1: Water-level monitor

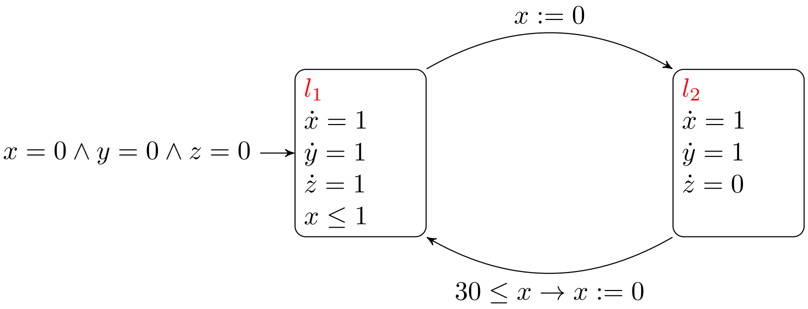


Figure 7.2: Leaking gas burner

- *For all edges, the* transision relation *is defined by a guarded set of nondeterministic assignments:*

$$\psi \Rightarrow \{x := [\alpha_x, \beta_x] \mid x \in \mathit{Var}\},$$

*where the guard $\psi$ is a linear formula and $\alpha_x, \beta_x$ are linear terms. For the transition relation $\mu \subseteq V^2$ we have*

$$(\nu, \nu') \in \mu \quad \textit{iff} \quad \nu \models \psi \wedge \forall x \in \mathit{Var}.\ \nu(\alpha_x) \leq \nu'(x) \leq \nu(\beta_x).$$

Figures 7.1 and 7.2 give two examples for linear hybrid automata.

The semantics of linear hybrid automata is given by the semantics of hybrid automata, specified by the following rules for discrete and time steps:

$$\frac{(l, a, \mu, l') \in \mathit{Edge} \quad (\nu, \nu') \in \mu \quad \nu' \in \mathit{Inv}(l')}{(l, \nu) \xrightarrow{a} (l', \nu')} \ \texttt{Rule}_{\texttt{discrete}}$$

$$\frac{\begin{array}{c} f \in Act(l) \quad f(0) = \nu \quad f(t) = \nu' \\ t \geq 0 \quad \forall 0 \leq t' \leq t. f(t') \in Inv(l) \end{array}}{(l, \nu) \xrightarrow{t} (l, \nu')} \ \texttt{Rule}_{\texttt{time}}$$

For time-deterministic hybrid automata the time-step rule can be simplified using the following predicate.

**Definition 7.3.** *For time-deterministic hybrid automata we define the* "time can progress" *predicate:*

$tcp_l[\nu](t)$

$$tcp_l[\nu](t) \quad \textit{iff} \quad \forall 0 \leq t' \leq t.\ f_l[\nu](t') \in Inv(l).$$

Thus for time-deterministic automata we can rewrite the time-step rule to

$$\frac{t \geq 0 \quad tcp_l[\nu](t)}{(l, \nu) \xrightarrow{t} (l, f_l[\nu](t))} \ \texttt{Rule}'_{\texttt{time}}$$

## 7.2 Forward Analysis

The reachability problem for linear hybrid automata is in general undecidable. However, bounded reachability is still decidable. Despite of undecidability, for the general rechability analysis of linear hybrid automata there exist incomplete algorithms. In this section we describe such a technique, a forward analysis approach based on fixed-point computation.

In general, forward analysis techniques start from the initial state set $R_0$ of a system, and compute the state set $R_1$ reachable from $R_0$ within one computation step. For the resulting set the same computation is repeated, i.e., the state set $R_2$ reachable in one transition step from $R_1$ is computed. The algorithm terminates if after a number of steps no new states can be reached, i.e., if $R_k \subseteq \bigcup_{i=0}^{k-1} R_i$ for some $k > 0$. Termination corresponds to finding the least fixed-point for the one-step (forward) reachability starting from the initial set. After termination we can check if all states in the determined reachable set satisfy the required property. Note that the computation may in general not terminate if the state space is infinite.

The one-step reachability for continuous steps is described by the following notion of forward time closure:

**Definition 7.4.** *We define the* forward time closure $\langle P \rangle_l^{\nearrow}$ *of* $P \subseteq V$ *at* $l \in Loc$ *as the set of valuations reachable from* $P$ *by letting time progress:*

$\langle P \rangle_l^{\nearrow}$

$$\nu' \in \langle P \rangle_l^{\nearrow} \quad \textit{iff} \quad \exists \nu \in P.\ \exists t \in \mathbb{R}_{\geq 0}.\ tcp_l[\nu](t) \wedge \nu' = f_l[\nu](t).$$

*We extend the definition to regions* $R = \cup_{l \in Loc}(l, R_l)$ *as follows:*

$\langle R \rangle_l^{\nearrow}$

$$\langle R \rangle^{\nearrow} = \cup_{l \in Loc}(l, \langle R_l \rangle_l^{\nearrow}).$$

*For the discrete steps, the corresponding one-step relation is formalized by postconditions:*
*We define the* postcondition $post_e[P]$ *of* $P$ *with respect to an edge* $e = (l, a, \mu, l')$ *as the set of valuations reachable from* $P$ *by* $e$*:*

$post_e[P]$

$$\nu' \in post_e[P] \quad \textit{iff} \quad \exists \nu \in P.\ (\nu, \nu') \in \mu.$$

$post[R]$

*An extension to regions $R = \cup_{l\in Loc}(l, R_l)$ is defined as follows:*

$$post[R] = \cup_{e=(l,a,\mu,l')\in Edge}(l', post_e[R_l]).$$

Note that, due to the $\tau$-transitions, $R \subseteq post[R]$. Similarly, due to time steps of duration 0 we have $R \subseteq \langle R\rangle^{\nearrow}$.

**Lemma 7.1.** *For all linear hybrid automata, if $P \subseteq V$ is a linear set of valuations, then for all $l \in Loc$ and $e \in Edge$, both $\langle P\rangle_l^{\nearrow}$ and $post_e[P]$ are linear sets of valuations.*

The set of states reachable in a finite number of steps from the initial state set form the reachable region of the automaton.

$(I, \mapsto^*)$

**Definition 7.5.** *Given a region $I \subseteq \Sigma$, the* reachable region $(I \mapsto^*) \subseteq \Sigma$ *of $I$ is the set of all states that are reachable from states in $I$:*

$$\sigma \in (I \mapsto^*) \quad \textit{iff} \quad \exists \sigma' \in I.\ \sigma' \to^* \sigma.$$

The following lemma states, that if the forward analysis procedure terminates, then the result, being the least fixed-point of the one-step relation, gives us the set of all reachable states.

**Lemma 7.2.** *Let $I = \cup_{l\in Loc}(l, I_l)$ be a region of the linear hybrid automaton $A$. The reachable region $(I, \mapsto^*) = \cup_{l\in Loc}(l, R_l)$ is the least fixed-point of the equation*

$$X = \langle I \cup post[X]\rangle^{\nearrow}$$

*or, equivalently, for all locations $l \in Loc$, the set $R_l$ of valuations is the least fixed-point of the set of equations*

$$X_l = \langle I_l \cup \bigcup_{e=(l',a,\mu,l)\in Edge} post_e[X_{l'}]\rangle_l^{\nearrow}.$$

**Example 7.1 (Example forward reachability computation).** *Consider the example automaton from Figure* **??** *and assume that bad states $(l, \nu)$ are characterized by $\nu(x) = \nu(y) + 2$, independently of the location. We represent the initial sets, activities, invariants, transition relations and the bad states by linear real arithmetic formulas as follows:*

$$\begin{array}{lcllcl}
Init_{\ell_1}(x,y) & = & x = 0 \wedge y = 0 & Init_{\ell_2}(x,y) & = & \mathit{false} \\
f^x_{\ell_1}(x,y)(t) & = & x + t & f^x_{\ell_2}(x,y)(t) & = & x \\
f^y_{\ell_1}(x,y)(t) & = & y & f^y_{\ell_2}(x,y)(t) & = & y + t \\
Inv_{\ell_1}(x,y) & = & x \le y + 1 & Inv_{\ell_2}(x,y) & = & y \le x + 1 \\
 & & & & & \\
\mu^x_{\ell_2\to\ell_1}(x,y) & = & x & \mu^x_{\ell_1\to\ell_2}(x,y) & = & x \\
\mu^y_{\ell_2\to\ell_1}(x,y) & = & y & \mu^y_{\ell_1\to\ell_2}(x,y) & = & y \\
 & & & & & \\
Bad_{\ell_1}(x,y) & = & x = y + 2 & Bad_{\ell_2}(x,y) & = & x = y + 2
\end{array}$$

*The forward reachability analysis computes the following state sets represented again as linear real arithmetic formulas:*

$$\begin{array}{lclcl}
R^0_{\ell_1}(x,y) & = & Init_{\ell_1}(x,y) \wedge Inv_{\ell_1}(x,y) & = & x = 0 \wedge y = 0 \wedge x \le y + 1 \\
R^0_{\ell_2}(x,y) & = & Init_{\ell_2}(x,y) \wedge Inv_{\ell_2}(x,y) & = & \mathit{false}
\end{array}$$

$$
\begin{array}{lcl}
R^1_{\ell_1}(x,y) & = & \mathcal{T}^+_{\ell_1}(R^0_{\ell_1}) \\
 & = & \exists x',y',t.\ R^0_{\ell_1}(x',y') \wedge t \geq 0 \wedge x = f^x_{\ell_1}(x',y')(t) \wedge y = f^y_{\ell_1}(x',y')(t) \wedge \mathit{Inv}_{\ell_1}(x,y) \\
 & = & \exists x',y',t.\ \underbrace{x' = 0}_{elim.\ x'} \wedge \underbrace{y' = 0}_{elim.\ y'} \wedge x' \leq y' + 1 \wedge t \geq 0 \wedge x = x' + t \wedge y = y' \wedge x \leq y + 1 \\
 & = & \exists t.\ 0 \leq 1 \wedge t \geq 0 \wedge \underbrace{x = t}_{elim.\ t} \wedge y = 0 \wedge x \leq y + 1 \\
 & = & x \geq 0 \wedge y = 0 \wedge x \leq y + 1 \\
R^1_{\ell_2}(x,y) & = & \mathcal{T}^+_{\ell_2}(R^0_{\ell_2}) \\
 & = & \exists x',y',t.\ R^0_{\ell_2}(x',y') \wedge t \geq 0 \wedge x = f^x_{\ell_2}(x',y')(t) \wedge y = f^y_{\ell_2}(x',y')(t) \wedge \mathit{Inv}_{\ell_2}(x,y) \\
 & = & \mathit{false}
\end{array}
$$

$$
\begin{array}{lcl}
R^2_{\ell_1}(x,y) & = & \mathcal{D}^+(R^1_{\ell_2}) \\
 & = & \exists x',y'.\ R^1_{\ell_2}(x',y') \wedge x = \mu^x_{\ell_2 \to \ell_1}(x',y') \wedge y = \mu^y_{\ell_2 \to \ell_1}(x',y') \wedge \mathit{Inv}_{\ell_1}(x,y) \\
 & = & \mathit{false} \\
R^2_{\ell_2}(x,y) & = & \mathcal{D}^+(R^1_{\ell_1}) \\
 & = & \exists x',y'.\ R^1_{\ell_1}(x',y') \wedge x = \mu^x_{\ell_1 \to \ell_2}(x',y') \wedge y = \mu^y_{\ell_1 \to \ell_2}(x',y') \wedge \mathit{Inv}_{\ell_2}(x,y) \\
 & = & \exists x',y'.\ x' \geq 0 \wedge y' = 0 \wedge x' \leq y' + 1 \wedge \underbrace{x = x'}_{elim.\ x'} \wedge \underbrace{y = y'}_{elim.\ y'} \wedge y \leq x + 1 \\
 & = & x \geq 0 \wedge y = 0 \wedge x \leq y + 1 \wedge y \leq x + 1
\end{array}
$$

$$
\begin{array}{lcl}
R^3_{\ell_1}(x,y) & = & \mathcal{T}^+_{\ell_1}(R^2_{\ell_1}) \\
 & = & \exists x',y',t.\ R^2_{\ell_1}(x',y') \wedge t \geq 0 \wedge x = f^x_{\ell_1}(x',y')(t) \wedge y = f^y_{\ell_1}(x',y')(t) \wedge \mathit{Inv}_{\ell_1}(x,y) \\
 & = & \mathit{false} \\
R^3_{\ell_2}(x,y) & = & \mathcal{T}^+_{\ell_2}(R^2_{\ell_2}) \\
 & = & \exists x',y',t.\ R^2_{\ell_2}(x',y') \wedge t \geq 0 \wedge x = f^x_{\ell_2}(x',y')(t) \wedge y = f^y_{\ell_2}(x',y')(t) \wedge \mathit{Inv}_{\ell_2}(x,y) \\
 & = & \exists x',y',t.\ x' \geq 0 \wedge \underbrace{y' = 0}_{elim.\ y'} \wedge x' \leq y' + 1 \wedge y' \leq x' + 1 \wedge t \geq 0 \wedge \\
 & & \underbrace{x = x'}_{elim.\ x'} \wedge y = y' + t \wedge y \leq x + 1 \\
 & = & \exists t.\ x \geq 0 \wedge x \leq 1 \wedge 0 \leq x + 1 \wedge t \geq 0 \wedge \underbrace{y = t}_{elim.\ t} \wedge y \leq x + 1 \\
 & = & x \geq 0 \wedge x \leq 1 \wedge y \geq 0 \wedge y \leq x + 1
\end{array}
$$

$$
\begin{array}{lcl}
R^4_{\ell_1}(x,y) & = & \mathcal{D}^+(R^3_{\ell_2}) \\
 & = & \exists x',y'.\ R^3_{\ell_2}(x',y') \wedge x = \mu^x_{\ell_2 \to \ell_1}(x',y') \wedge y = \mu^y_{\ell_2 \to \ell_1}(x',y') \wedge \mathit{Inv}_{\ell_1}(x,y) \\
 & = & \exists x',y'.\ x' \geq 0 \wedge x' \leq 1 \wedge y' \geq 0 \wedge y' \leq x' + 1 \wedge \underbrace{x = x'}_{elim.\ x'} \wedge \underbrace{y = y'}_{elim.\ y'} \wedge x \leq y + 1 \\
 & = & x \geq 0 \wedge x \leq 1 \wedge y \geq 0 \wedge y \leq x + 1 \wedge x \leq y + 1 \\
R^4_{\ell_2}(x,y) & = & \mathcal{D}^+(R^3_{\ell_1}) \\
 & = & \exists x',y'.\ R^3_{\ell_1}(x',y') \wedge x = \mu^x_{\ell_1 \to \ell_2}(x',y') \wedge y = \mu^y_{\ell_1 \to \ell_2}(x',y') \wedge \mathit{Inv}_{\ell_2}(x,y) \\
 & = & \mathit{false}
\end{array}
$$

$$
\begin{array}{lcl}
R^5_{\ell_1}(x,y) & = & \mathcal{T}^+_{\ell_1}(R^4_{\ell_1}) \\
 & = & \exists x',y',t.\ R^4_{\ell_1}(x',y') \wedge t \geq 0 \wedge x = f^x_{\ell_1}(x',y')(t) \wedge y = f^y_{\ell_1}(x',y')(t) \wedge \mathit{Inv}_{\ell_1}(x,y) \\
 & = & \exists x',y',t.\ x' \geq 0 \wedge x' \leq 1 \wedge y' \geq 0 \wedge y' \leq x' + 1 \wedge x' \leq y' + 1 \wedge \\
 & & t \geq 0 \wedge \underbrace{x = x' + t}_{elim.\ x'} \wedge \underbrace{y' = y}_{elim.\ y'} \wedge x \leq y + 1 \\
 & = & \exists t.\ x - t \geq 0 \wedge x - t \leq 1 \wedge y \geq 0 \wedge y \leq x - t + 1 \wedge x - t \leq y + 1 \wedge
\end{array}
$$

$$
\begin{array}{rcl}
 & & t \geq 0 \wedge x \leq y+1 \\
 & = & \exists t.\ \ \underbrace{x \geq t}_{upper\ bound} \wedge \underbrace{x-1 \leq t}_{lower\ bound} \wedge y \geq 0 \wedge \underbrace{t \leq x-y+1}_{upper\ bound} \wedge \underbrace{x-y-1 \leq t}_{lower\ bound} \wedge \\
 & & \underbrace{t \geq 0}_{lower\ bound} \wedge x \leq y+1 \\
 & = & x-1 \leq x \wedge x-1 \leq x-y+1 \wedge x-y-1 \leq x \wedge x-y-1 \leq x-y+1 \wedge \\
 & & 0 \leq x \wedge 0 \leq x-y+1 \wedge y \geq 0 \wedge x \leq y+1 \\
 & = & y \leq 2 \wedge 0 \leq x \wedge y \leq x+1 \wedge y \geq 0 \wedge x \leq y+1 \\
R^5_{\ell_2}(x,y) & = & \mathcal{T}^+_{\ell_2}(R^4_{\ell_2}) \\
 & = & \exists x', y', t.\ R^4_{\ell_2}(x',y') \wedge t \geq 0 \wedge x = f^x_{\ell_2}(x',y')(t) \wedge y = f^y_{\ell_2}(x',y')(t) \wedge Inv_{\ell_2}(x,y) \\
 & = & false
\end{array}
$$

*This computation sequence will not terminate, since each new iteration reaches some new states, but none of the computed sets intersect with the sets of bad states (which are actually not reachable).*

## 7.3 Backward Analysis

There is a similar backward approach for the fixed-point-based reachability analysis of linear hybrid automata. Instead of starting from the initial set and computing successors like in the forward approach, the backward search starts from a target set, defined as the set of states violating the property to be proved, and computes stepwise predecessors. The algorithm terminates if it finds the least fixed-point for the reversed one-step relation, thereby determining the set of states from which the target set can be reached. If the intersection of the resulting set with the initial set is empty, the property holds, otherwise the property does not hold.

Analogously to the forward time closure for the time steps and the postcondition for discrete steps in the forward approach, we define for the reversed steps a backward time closure for time steps and a precondition for discrete steps.

$\langle P \rangle_l^{\swarrow}$ **Definition 7.6.** *We define the* backward time closure $\langle P \rangle_l^{\swarrow}$ *of* $P \subseteq V$ *at* $l \in Loc$ *as the set of valuations from which it is possible to reach a valuation in* $P$ *by letting time progress:*

$$\nu' \in \langle P \rangle_l^{\swarrow} \quad \textit{iff} \quad \exists \nu \in P.\ \exists t \in \mathbb{R}_{\geq 0}.\ tcp_l[\nu'](t) \wedge \nu = f_l[\nu'](t).$$

$\langle R \rangle^{\swarrow}$ *We extend the definition to regions* $R = \cup_{l \in Loc}(l, R_l)$ *as follows:*

$$\langle R \rangle^{\swarrow} = \cup_{l \in Loc}(l, \langle R_l \rangle_l^{\swarrow}).$$

$pre_e[P]$ *We define the* precondition $pre_e[P]$ *of* $P$ *with respect to an edge* $e = (l, a, \mu, l')$ *as the set of valuations from which it is possible to reach a valuation from* $P$ *by* $e$*:*

$$\nu' \in pre_e[P] \quad \textit{iff} \quad \exists \nu \in P.\ (\nu', \nu) \in \mu.$$

$pre[R]$ *For regions* $R = \cup_{l \in Loc}(l, R_l)$ *we define*

$$pre[R] = \cup_{e=(l',a,\mu,l) \in Edge}(l', pre_e[R_l]).$$

Note that, due to the $\tau$-transitions, $R \subseteq pre[R]$. Similarly, due to time steps of duration 0 we have $R \subseteq \langle R \rangle^{\swarrow}$.

**Lemma 7.3.** *For all linear hybrid automata, if $P \subseteq V$ is a linear set of valuations, then for all $l \in Loc$ and $e \in Edge$, both $\langle P \rangle_l^{\swarrow}$ and $pre_e[P]$ are linear sets of valuations.*

For a target state set we define its initial region as the set of states from which the target set is reachable.

**Definition 7.7.** *Given a region $R \subseteq \Sigma$, the* initial region $(\mapsto^* R) \subseteq \Sigma$ *of $R$ is the set of all states from which a state in $R$ is reachable:* $(\mapsto^* R)$

$$\sigma \in (\mapsto^* R) \quad \textit{iff} \quad \exists \sigma' \in R.\ \sigma \rightarrow^* \sigma'.$$

The following lemma states that if the backward algorithm terminates, it determines the states from which the target region is reachable.

**Lemma 7.4.** *Let $R = \cup_{l \in Loc}(l, R_l)$ be a region of the linear hybrid automaton $A$. The initial region $(\mapsto^* R) = \cup_{l \in Loc}(l, I_l)$ of $R$ is the least fixed-point of the equation*

$$X = \langle R \cup pre[X] \rangle^{\swarrow}$$

*or, equivalently, for all locations $l \in Loc$, the set $I_l$ of valuations is the least fixed-point of the set of equations*

$$X_l = \langle R_l \cup \bigcup_{e=(l,a,\mu,l') \in Edge} pre_e[X_{l'}] \rangle_l^{\swarrow}.$$

**Example 7.2 (Example backward reachability computation).** *Consider again the example automaton from Figure* **??** *and the same representations of intial sets, activities, invariants, transition relations and bad states as in Example 7.1.*

*The backward reachability computation generates the following set representations:*

$$\begin{array}{lclclcl}
R^0_{\ell_1}(x,y) & = & Bad_{\ell_1}(x,y) \wedge Inv_{\ell_1}(x,y) & = & x = y+2 \wedge x \leq y+1 & = & false \\
R^0_{\ell_2}(x,y) & = & Bad_{\ell_2}(x,y) \wedge Inv_{\ell_2}(x,y) & = & x = y+2 \wedge y \leq x+1 & &
\end{array}$$

$$\begin{array}{lcl}
R^1_{\ell_1}(x,y) & = & \mathcal{T}^-_{\ell_1}(R^0_{\ell_1}) \\
& = & \exists x', y', t.\ R^0_{\ell_1}(x',y') \wedge t \geq 0 \wedge x' = f^x_{\ell_1}(x,y)(t) \wedge y' = f^y_{\ell_1}(x,y)(t) \wedge Inv_{\ell_1}(x,y) \\
& = & false \\
R^1_{\ell_2}(x,y) & = & \mathcal{T}^-_{\ell_2}(R^0_{\ell_2}) \\
& = & \exists x', y', t.\ R^0_{\ell_2}(x',y') \wedge t \geq 0 \wedge x' = f^x_{\ell_2}(x,y)(t) \wedge y' = f^y_{\ell_2}(x,y)(t) \wedge Inv_{\ell_2}(x,y) \\
& = & \exists x', y', t.\ x' = y'+2 \wedge t \geq 0 \wedge \underbrace{x' = x+t}_{elim.\ x'} \wedge \underbrace{y' = y}_{elim.\ y'} \wedge x \leq y+1 \\
& = & \exists t.\ \underbrace{x+t = y+2}_{elim.\ t} \wedge t \geq 0 \wedge x+t \leq y+1 \\
& = & y - x + 2 \geq 0 \wedge x + y - x + 2 \leq y+1 \\
& = & false
\end{array}$$

*Already the first iteration does not yield any new state, i.e., the algorithm terminates. Since none of the computed stes intersects with the initial state sets, no bad states can be reached from any initial state. (Note that it is even not possible to reach a bad state from any good state.)*

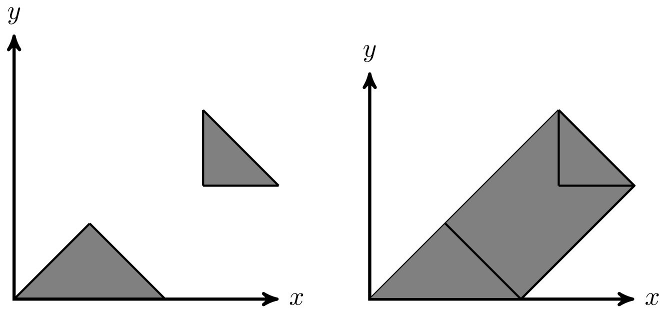


Figure 7.3: Two sets (left) and their convex hull (right)

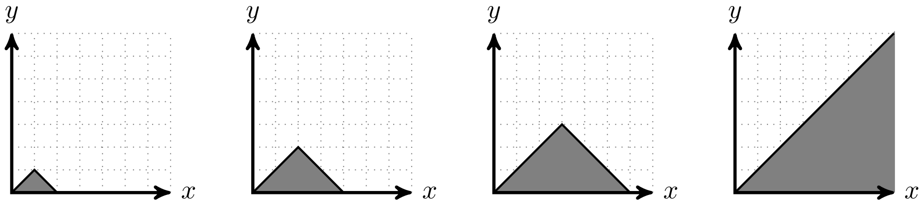


Figure 7.4: A sequence of three sets (left three pictures) and their widening (right)

## 7.4 Approximate Analysis

If the (forward or backward) iterative techniques does not converge, we can compute *over-approximations* of the sets

- $(I \mapsto^*)$ of states which are reachable from the initial states $I$ (forward analysis), or
- $(\mapsto^* R)$ of states from which the region $R$ is reachable (backward analysis).

Below we discuss two approaches for over-approximation: the first one is based on building convex hulls, and the second one is a widening technique.

1. Instead of computing the *union* of sets, we can compute their *convex hull*, i.e., the smallest convex polyhedron containing the operands of the union (see Figure 7.3). Though this set over-approximates the exact result, it may help the algorithms to terminate. On the one hand, if with the over-approximation we can show the correctness of the property we want to prove, then we are happy with the result: if the property holds even for the over-approximation then if holds also for the over-approximated reachable set. On the other hand, if the proof fails, then, due to the over-approximation, it does not mean that the property does not hold: those states of the over-approximation that violate the property may lie outside of the exact, over-approximated set and are thus perhaps not reachable. In this case we must try to find a more accurate over-approximation.

2. To enforce the convergence of iterations, we can apply a *widening technique*. The basic idea is to extrapolate the limit of the state set sequence occurring in the non-terminating fixed-point computation. The standard widening algorithm applies the widening for at least one location in each *loop* of the hybrid automaton graph. Figure 7.4 illustrates the widening technique.

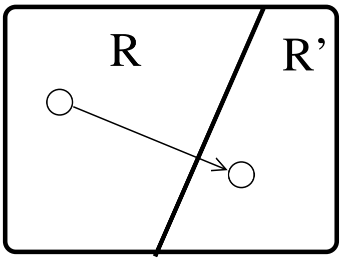


Figure 7.5: The next relation $\mapsto$ on regions

## 7.5 Minimization

In this section we discuss another approach called *minimization* for the analysis of linear hybrid automata, based on abstraction and abstraction refinement. We introduce a forward method but it is also possible to define it for a backward search.

Assume a linear hybrid automaton and a safety property whose validity we want to check. The property divides the state space of the hybrid automaton into a set of "good" states that satisfy the property and a set of "bad" states that violate it. Let $R_{bad}$ denote the set of violating states. To check the validity of the property we check if a state from $R_{bad}$ is reachable.

The abstraction is based on partitioning the state space of a linear hybrid automaton into a finite set $\Pi = \{R_{bad}, R_1, \ldots, R_n\}$ of regions with $R_{bad} \cap R_i = \emptyset$ for all $1 \leq i \leq n$, $R_i \cap R_j = \emptyset$ for all $1 \leq i < j \leq n$, and $\Sigma = R_{bad} \cup \bigcup_{i=1}^{n} R_i$. Each such partitioning induces a LSTS being an abstraction of the linear hybrid automaton. The abstract states of the LSTS are the regions of the partitioning. The regions containing at least one concrete initial state are the abstract initial states. There is a transition from a region $R$ to a region $R'$ of the partitioning, denoted by $R \mapsto R'$, iff from at least one state in $R$ at least one state in $R'$ is reachable in one step. Since we are only interested in the reachability of bad states, we define no successors for $R_{bad}$. The abstract transitions are formalized as follows:

**Definition 7.8.** *The next relation $\mapsto$ on regions is defined by* $\mapsto$

$$R \mapsto R' \quad \text{iff} \quad R \neq R_{bad} \;\wedge\; \exists \sigma \in R.\ \exists \sigma' \in R'.\ \sigma \to \sigma'.$$

Figure 7.5 illustrates the next relation.

Such an abstraction in general over-approximates the behaviour of the concrete system: For each reachable state of the concrete system the region of the abstraction that contains that state is also reachable. However, there may be regions reachable in the abstraction that contain no states reachable in the concrete system.

That implies on the one hand, that if $R_{bad}$ is not reachable in the abstraction then the property holds for the concrete system. But on the other hand, from the reachability of $R_{bad}$ in the abstraction we cannot conclude that the property does not hold for the original system. However, we can define a sufficient condition under that the second implication also holds, i.e., a condition that assures that $R_{bad}$ is reachable in the abstraction if and only if the concrete system violates the property. This condition is that all regions reachable in the abstraction have at least one state reachable in the concrete system. The minimization algorithm starts with an initial partitioning and splits regions of the partitioning iteratively until it satisfies that sufficient condition. Note that, since the reachability problem for linear hybrid automata is not decidable, the refinement loop does not always terminate. But in case it terminates, the abstraction is finite, and we can answer the reachability question.

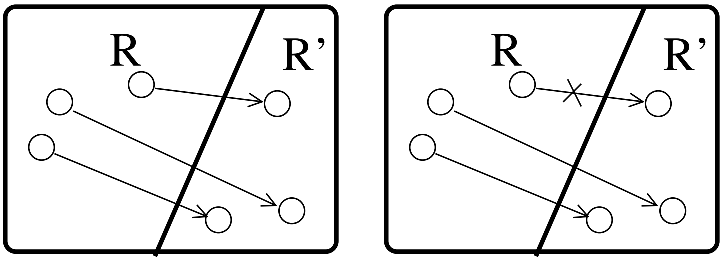


Figure 7.6: Stability of regions: a stable region (left) and a non-stable one (right)

How can we be sure that a region $R$ reachable in the abstraction contains at least one state reachable in the concrete system? First we only know that all initial regions contain at least one initial state by definition. Now assume a reachable region $R$ that contains at least one state $\sigma \in R$ reachable in the concrete system, and assume a successor region $R'$ of $R$ with $R \mapsto R'$. From $R \mapsto R'$ we conclude that there is a state in $R$ with a successor state in $R'$, however, we do not know if this state is $\sigma$. But, if *all* states in $R$ have a successor state in $R'$, then also $\sigma$ has a successor state $\sigma' \in R'$, and from the reachability of $\sigma$ together with $\sigma \to \sigma'$ we can conclude that there is at least one reachable state in $R'$.

**Definition 7.9.** *Let $\Pi$ be a partitioning of the state space $\Sigma$ and let $R, R' \in \Pi$. The region $R$ is called* stable for $R'$ *iff*

$$R \mapsto R' \quad \textsf{implies} \quad \forall \sigma \in R.\ \{\sigma\} \mapsto R'.$$

*We call $R$ stable iff it is stable for all regions in $\Pi$. We call $\Pi$ stable iff all reachable regions of $\Pi$ are stable.*

Figure 7.6 illustrates the stability of regions.

Now we come to the algorithm as specified by Figure 7.7. The set of initial states of the concrete system is denoted by $I$, and $R_{bad}$ is the set of "bad" states. The algorithm stores the current partitioning in $\Pi$. Initially there are two regions in the patitioning: the region $R_{bad}$ contains all "bad" states and the region $\Sigma \backslash R_{bad}$ the "good" states.

The algorithm uses two sets *reach* and *completed*. In the set *reach* we store those reachable regions of the current partitioning for which we know that they contain at least one concrete state that is reachable in the concrete system. In the set *completed* $\subseteq$ *reach* we store regions from which we know that their successor regions are all in *reach*, i.e., regions that currently cannot be used to derive further in the concrete system reachable regions. Initially, *reach* contains those regions of the initial partitioning that contain at least one concrete intial state. The set *completed* is initially empty.

In each refinement step we determine a reachable region $R \in$ *reach* from that we already know that it has at least one reachable state, but we do not yet know if all of its successor regions contain reachable states, i.e., such that $R$ is not in *completed*. For all those successor regions of $R$ for which $R$ is stable we can conclude that also they contain at least one reachable state, thus we put them into the *reach* set.

If, after that update, all successor regions of $R$ are in *reach*, i.e., they all have at least one reachable state, then we put $R$ into the *completed* set.

Otherwise, if there is still a successor region $R' \notin$ *reach* of $R$ then $R$ is not stable for $R'$. We use such an $R'$, found at last, to split $R$ into two parts, one containing all states with a successor in $R'$ and a second part containing the rest. The splitting of a region is formalized by

```
$minimize(\Sigma, R_{bad})$ {
  $\Pi := \{R_{bad}, \Sigma \setminus R_{bad}\};$ $reach := \{R \in \Pi | R \cap I \neq \emptyset\};$ $completed := \emptyset;$
  **while** $(R_{bad} \notin reach \wedge reach \neq completed)$ {
    choose $R \in (reach \setminus completed);$ $S := \emptyset;$
    **for each** $(R' \in (\Pi \setminus reach)$ *with* $R \mapsto R')$ {
      $reach' := split(\Pi, R, R');$
      **if** $(reach' = R)$ **then** $reach := reach \cup \{R'\};$
      **else** $S := reach';$
    }
    **if** $(S = \emptyset)$ **then** $completed := completed \cup \{R\};$
    **else** {
      $\Pi := (\Pi \setminus \{R\}) \cup S;$
      $reach := (reach \setminus \{R\}) \cup \{S_i \in S \mid S_i \cap I \neq \emptyset\};$
      $completed := completed \setminus \{R' \in \Pi \mid \exists S_i \in (S \setminus reach).\ R' \mapsto S_i\};$
    }
  }
  **return** $R_{bad} \in reach;$
}
```

Figure 7.7: The minimization algorithm

the following definition:

**Definition 7.10.**

split$(\Pi, R, R')$

$$\mathsf{split}(\Pi, R, R') \quad := \quad \begin{cases} \{R'', R \setminus R''\} & \text{if } R'' = \mathsf{pre}[< R' >^{\swarrow}] \cap R \wedge R'' \neq \emptyset \wedge R'' \neq R, \\ \{R\} & \text{otherwise.} \end{cases}$$

Figure 7.8 illustrates the splitting mechanism.

We split $R$ according to the splitting result remembered in $S = \{S_1, S_2\}$, and update the partitioning. The *reach* set gets updated in that we remove $R$ and add $S_i$, $i = 1, 2$, if they contain concrete initial states. Note that, though we know that there is a concrete state either in $S_1$ or in $S_2$ that is reachable in the concrete system, we do not know which of both sets contains it. Thus we can add $S_1$ or $S_2$ to *reach* only if they contain concrete initial states. Note also that all other elements $R' \neq R$ in *reach* can stay in the set. Previous predecessors of $R$ are now predecessors of $S_1$ and/or $S_2$. For such predecessors that are in *completed* we check if still all of their successors are in *reach*, and remove them from *completed* if it is not the case. All other regions in *completed* remain in the set.

We observe that, since "bad" regions do not have outgoing transitions in the abstract LSTS, they are never split. Thus there is always a single "bad" region in the partitioning.

Before each iteration we check if one of the termination conditions hold: If $R_{bad} \in reach$ then the system violates the property. Otherwise, if $R_{bad} \notin reach$ but $reach = completed$ then $R_{bad}$ is not reachable in the abstraction, and the property holds.

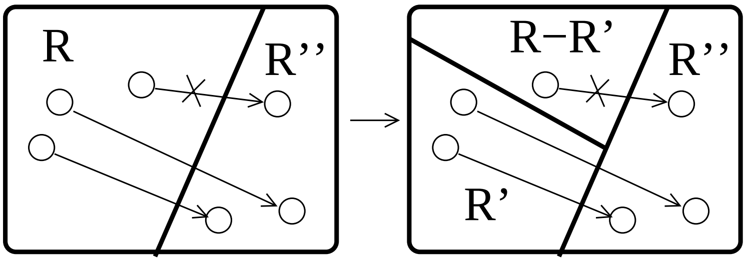


Figure 7.8: The splitting of regions

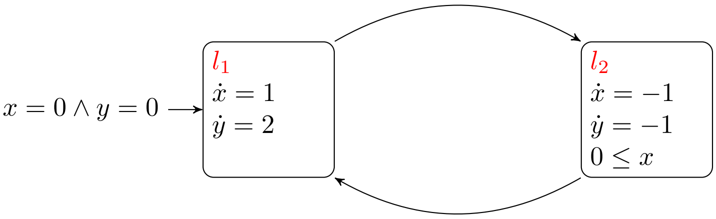


Figure 7.9: Leaking gas burner

Note that if the regions $R_{bad}$ and $I$ are linear, all regions that are constructed by the procedure are linear.

**Lemma 7.5.** *The procedure in Figure 7.7 returns TRUE iff* $I \mapsto^* R_{bad}$.

**Example 7.3.** *Assume the linear hybrid automaton shown in Figure 7.9. We want to prove that* $0 \leq y$ *always holds.*

*We have*

$$\begin{array}{rcl} R_{bad} & = & (\ell_1, y < 0) \cup (\ell_2, 0 \leq x \wedge y < 0) \\ R_1 & = & (\ell_1, 0 \leq y) \cup (\ell_2, 0 \leq x \wedge 0 \leq y) \end{array}$$

*The algorithm initializes*

$$\begin{array}{rcl} \Pi & = & \{R_{bad}, R_1\} \\ \mathit{reach} & = & \{R_1\} \\ \mathit{completed} & = & \emptyset \ . \end{array}$$

*Since* $R_{bad} \notin \mathit{reach}$ *and* $\mathit{reach} \neq \mathit{completed}$ *the main loop is entered. We choose the only element* $R_1 \in \mathit{reach}$. *Its only successor region is* $R_{bad}$. *We first compute the time predecessor of* $R_{bad}$:

$$\begin{array}{rcl} \langle R_{bad} \rangle^{\swarrow} & = & \langle (\ell_1, y < 0) \cup (\ell_2, 0 \leq x \wedge y < 0) \rangle^{\swarrow} \\ & = & \langle (\ell_1, y < 0) \rangle^{\swarrow} \cup \langle (\ell_2, 0 \leq x \wedge y < 0) \rangle^{\swarrow} \end{array}$$

*To compute* $\langle (\ell_1, y < 0) \rangle^{\swarrow}$ *assume a time step resulting in a state from* $(\ell_1, y < 0)$. *Then the control is in* $\ell_1$ *also before the time step. For the valuation, if* $x$ *and* $y$ *denote the values before* *the time step, then after the time step the values change to $x+t$ and $y+2t$ for some $0 \leq t$, and we know that $y+2t<0$. We have to eliminate $t$ from the equation set*

$$0 \leq t \wedge y+2t < 0,$$

*i.e.,*

$$0 \leq t \wedge t < -y/2$$

*which yield $\langle(\ell_1, y<0)\rangle^{\swarrow} = (\ell_1, y<0)$.*

*To compute $\langle(\ell_2, 0 \leq x \wedge y < 0)\rangle^{\swarrow}$ assume a time step resulting in a state from $(\ell_2, 0 \leq x \wedge y < 0)$. Then before the time step control is in $\ell_2$. Let again $x$ and $y$ denote the variable values before the time step. The time step changes the values to $x-t$ and $y-t$ for some $0 \leq t$. Due to the invariant $0 \leq x$ and $0 \leq x-t$, and since the target state should be from $R_{bad}$ we have $y-t<0$. Eliminating $t$ from the equation system*

$$0 \leq x \wedge 0 \leq x-t \wedge 0 \leq t \wedge y-t < 0,$$

*i.e.,*

$$0 \leq x \wedge t \leq x \wedge 0 \leq t \wedge y < t,$$

*we get $0 \leq x \wedge y < x$. Thus $\langle(\ell_2, 0 \leq x \wedge y < 0)\rangle^{\swarrow} = (\ell_2, 0 \leq x \wedge y < x)$.*

*Collecting the above information, $\langle R_{bad}\rangle^{\swarrow} = (\ell_1, y<0) \cup (\ell_2, 0 \leq x \wedge y < x)$.*

*Now we compute the discrete step predecessor of this set.*

$$\begin{array}{ll}
\mathit{pre}\,[\langle R_{bad}\rangle^{\swarrow}] & = \\
\mathit{pre}\,[(\ell_1, y<0) \cup (\ell_2, 0 \leq x \wedge y < x)] & = \\
\underbrace{(\ell_1, y<0)}_{\tau_{\ell_1}} \cup \underbrace{(\ell_2, 0 \leq x \wedge y < x)}_{\tau_{\ell_2}} \cup \underbrace{(\ell_2, 0 \leq x \wedge y < 0)}_{\textit{edge from } \ell_2 \textit{ to } \ell_1} \cup \underbrace{(\ell_1, 0 \leq x \wedge y < x)}_{\textit{edge from } \ell_1 \textit{ to } \ell_2} & = \\
(\ell_1, y<0 \vee 0 \leq y < x) \cup (\ell_2, (0 \leq x \wedge y < 0) \vee (0 \leq y < x)) &
\end{array}$$

*The intersection of this predecessor set with $R_1$ yields*

$$\begin{array}{rcl}
\mathit{pre}\,[\langle R_{bad}\rangle^{\swarrow}] \cap R_1 & = & [(\ell_1, y<0 \vee 0 \leq y < x) \cup (\ell_2, (0 \leq x \wedge y < 0) \vee (0 \leq y < x))] \cap \\
 & & [(\ell_1, 0 \leq y) \cup (\ell_2, 0 \leq x \wedge 0 \leq y)] \\
 & = & (\ell_1, 0 \leq y < x) \cup (\ell_2, 0 \leq y < x) \\
 & =: & R_2.
\end{array}$$

*We define*

$$R_3 := R_1 \backslash R_2 = (\ell_1, 0 \leq x \leq y) \cup (\ell_2, 0 \leq x \wedge 0 \leq y \wedge x \leq y).$$

*Thus we have split$(\Pi, R_1, R_{bad}) = \{R_2, R_3\}$. The corresponding updates result in*

$$\begin{array}{rcl}
\Pi & = & \{R_{bad}, R_2, R_3\} \\
\mathit{reach} & = & \{R_3\} \\
\mathit{completed} & = & \emptyset
\end{array}$$

*In the next iteration the termination conditions are still not met thus we execute the loop once more. For $R_3 \in$ reach we have no successor regions, thus the region does not get split and the update results in*

$$\begin{array}{rcl}
\Pi & = & \{R_{bad}, R_2, R_3\} \\
\mathit{reach} & = & \{R_3\} \\
\mathit{completed} & = & \{R_3\}.
\end{array}$$

*In the next iteration we detect that the termination condition* $reach = completed$ *holds.*

*Since* $R_{bad} \notin reach$*, the algorithm returns that the property holds.*

The minimization of linear hybrid automata is a special case of a more general approach frequently used for the reachability analysis of general hybrid systems. The general approach defines an initial partitioning of the state space and refines it by region splitting until it becomes fine enough to prove or violate the requested safety property. The different instances of this general approach use different methods to determine the regions to be split and the splitting itself.

## Exercises

Chapter 8

# General Hybrid Automata

In the previous chapter we have seen an approach for the reachability analysis of hybrid systems with linear behavior based on fixed point computation. There we represented the reachable sets by conjunctions of linear constraints. In this chapter we discuss the general language of *hybrid automata* and corresponding reachability analysis techniques, and especially other representation techniques for their state sets.

## 8.1 Syntax and Semantics of Hybrid Automata

In an LTS the values of the variables may change instantaneously by taking a *discrete* transition from one location to another. Hybrid automata extend LTSs: Additionally to such discrete state changes, while control stays in a location, times passes by, and the values of variables change *continuously* according to some continuous functions. The combination of the discrete and the continuous behaviour leads to the term “hybrid”.

**Definition 8.1 (Syntax of hybrid automata).** *A hybrid automaton $\mathcal{H}$ is a tuple $(\mathit{Loc}, \mathit{Var}, \mathit{Con}, \mathit{Lab}, \mathit{Edge},$ $\mathcal{H}$ $\mathit{Act}, \mathit{Inv}, \mathit{Ini}$ where*

- *$(\mathit{Loc}, \mathit{Var}, \mathit{Con}, \mathit{Lab}, \mathit{Edge}, \mathit{Init})$ is an LTS with real-valued variables $\mathit{Var}$, $V$ the set of all valuations $\nu : \mathit{Var} \to \mathbb{R}$, and $\Sigma = \mathit{Loc} \times V$ the set of states,*
- *$\mathit{Act}$ is a function assigning a set of activities $f : \mathbb{R}_{\geq 0} \to V$ to each location which are time-invariant meaning that $f \in \mathit{Act}(l)$ implies $(f+t) \in \mathit{Act}(l)$ where $(f+t)(t') = f(t+t')$ for all $t' \in \mathbb{R}_{\geq 0}$, and* *Act*
- *a function $\mathit{Inv}$ assigning an invariant $\mathit{Inv}(l) \subseteq V$ to each location $l \in \mathit{Loc}$.* *Inv*

Compared to LTS, we have two new components: the activities and the invariants attached to the locations. The activities describe the continuous state changes in the locations when time passes by. The invariants restrict this behaviour such that time can evolve only as long as the invariant of the current location is satisfied. The control must leave the location before the invariant gets violated using a discrete transition. Also entering a location by a discrete step is only possible if the target location’s invariant is satisfied after the step.

The execution of a hybrid automaton starts in a state $\sigma_0 = (\ell_0, \nu_0) \in \mathit{Init}$ from the initial set. The invariant $\mathit{Inv}(\ell_0)$ of the initial location $\ell_0$ must be satisfied by the initial valuation $\nu_0$, i.e., $\nu_0 \in \mathit{Inv}(\ell_0)$ must hold. Now two things can happen:

1. Time can pass by in the current location $\ell_0$, and the values of the variables evolve according to a function $f : \mathbb{R}_{\geq 0} \to V$ from $\mathit{Act}(\ell_0)$. The function $f$ must satisfy $f(0) = \nu_0$, i.e., it

assigns the initial valuation to the time point 0. After $t$ time units the variables' values are given by $\nu_1 = f(t)$, i.e., the system reaches the state $(\ell_0, \nu_1)$.

However, the control may stay in $\ell_0$ only as long as the invariant $\mathit{Inv}(\ell_0)$ of $\ell_0$ is satisfied. I.e., $t$ time can pass by only if $\forall 0 \leq t' \leq t$ we have $f(t) \in \mathit{Inv}(\ell_0)$.

2. A discrete state change can happen if there is an enabled edge from $\ell_0$, i.e., if there is a $(\ell_0, a, \mu, \ell_1) \in \mathit{Edge}$ and a valuation $\nu_1 \in V$ such that $(\nu_0, \nu_1) \in \mu$. The invariant of the target location must be satisfied after the step, i.e., $\nu_1 \in \mathit{Inv}(\ell_1)$ must hold.

From the state resulting from such a time or discrete step the system can again take either a time or a discrete step as described above.

**Definition 8.2 (Semantics of hybrid automata).** *The semantics of a hybrid automaton $\mathcal{H} = (\mathit{Loc}, \mathit{Var}, \mathit{Con}, \mathit{Lab}, \mathit{Edge}, \mathit{Act}, \mathit{Inv}, \mathit{Init})$ is given by an operational semantics consisting of two rules, one for the discrete instantaneous steps and one for the continuous time steps.*

1. *Discrete step semantics*

$$\frac{(l, a, (\nu, \nu'), l') \in \mathit{Edge} \quad \nu' \in \mathit{Inv}(l')}{(l, \nu) \xrightarrow{a} (l', \nu')}\ \mathtt{Rule}_{\mathtt{discrete}}$$

2. *Time step semantics*

$$\frac{f \in \mathit{Act}(l) \quad f(0) = \nu \quad f(t) = \nu' \quad t \geq 0 \quad f([0, t]) \subseteq \mathit{Inv}(l)}{(l, \nu) \xrightarrow{t} (l, \nu')}\ \mathtt{Rule}_{\mathtt{time}}$$

*An* execution step

$$\rightarrow \quad = \quad \xrightarrow{a} \cup \xrightarrow{t}$$

$\pi$ *of $\mathcal{H}$ is either a discrete or a time step. A* path *(or run or excecution) $\pi$ of $\mathcal{H}$ is a sequence $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\nu_0 \in \mathit{Inv}(\ell_0)$ and $\sigma_i \rightarrow \sigma_{i+1}$ for all $i \geq 0$. We use $\Pi_{\mathcal{H}}(\sigma)$ (or short $\Pi(\sigma)$) to denote the set of all paths of $\mathcal{H}$ starting in $\sigma$. A state $\sigma$ of $\mathcal{H}$ is* reachable *iff there is a run of $\mathcal{H}$ starting in an initial state of $\mathcal{H}$ and leading to $\sigma$.*

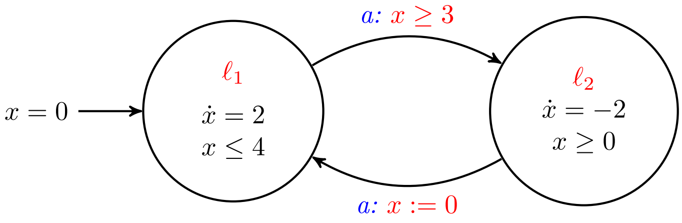
As it is the case for LTS, the operational semantics of hybrid automata define their induced state transition system. In the hybrid setting the set of reachables states is in general uncountable, as time progress leads to continuous behaviour.

Usually, the activities of a hybrid automaton are given implicitly by differential equations, the activities being their solutions. E.g., $\dot{x} = 1$ specifies a set of activities $f : \mathbb{R}_{\geq 0} \rightarrow V$ with $f(t)(x) = t + c$ for some $c \in \mathbb{R}$ being the value of $x$ at time point 0.

Furthermore, valuation sets like the invariants of the locations are usually specified by formulae of the first-order logic over the reals (without quantifiers). E.g., $x > 0$ specifies the valuation set $\{\nu \in V | \nu(x) > 0\}$.

Finally, similarly to LTSs, also hybrid automata are often given in a graphical representation. We illustate the modeling by hybrid automata on our previous examples of the bouncing ball, the thermostat, and the water-level monitor. In the graphical representations in the following we omit the $\tau$-transitions, non-synchronizing labels, trivial invariants, etc..

**Example 8.1.** *Assume the following graphical visualization of a hybrid automaton:*



*The formal definition is as follows:*

- $Loc = \{\ell_1, \ell_2\}$,
- $Var = \{x\}$,
- $Con(\ell_1) = Con(\ell_2) = \{x\}$,
- $Lab = \{\tau, a\}$,
- $Edge =$

$$\begin{aligned}\{\quad & (\ell_1,\ a,\ \{(\nu,\nu') \in V^2 \mid \nu(x) \geq 3 \wedge \nu'(x) = \nu(x)\},\ \ell_2),\\ & (\ell_2,\ a,\ \{(\nu,\nu') \in V^2 \mid \nu'(x) = 0\},\ \ell_1),\\ & (\ell_1,\ \tau,\ \{(\nu,\nu') \in V^2 \mid \nu = \nu'\},\ \ell_1),\\ & (\ell_2,\ \tau,\ \{(\nu,\nu') \in V^2 \mid \nu = \nu'\},\ \ell_2)\ \},\end{aligned}$$

- $Act(\ell_1) = \{f : \mathbb{R}_{\geq 0} \to V \mid \exists c \in \mathbb{R}. \forall t \in \mathbb{R}_{\geq 0}. f(t)(x) = 2t + c\}$,
  $Act(\ell_2) = \{f : \mathbb{R}_{\geq 0} \to V \mid \exists c \in \mathbb{R}. \forall t \in \mathbb{R}_{\geq 0}. f(t)(x) = -2t + c\}$,
- $Inv(\ell_1) = \{\nu \in V \mid \nu(x) \leq 4\}$,
  $Inv(\ell_2) = \{\nu \in V \mid \nu(x) \geq 0\}$,
- $Init = \{(\ell_1, \nu) \in \Sigma \mid \nu(x) = 0\}$.

*Note that the activity sets for both locations are time-invariant. The instances of the discrete rule of the semantics for the two non-$\tau$ discrete transitions are:*

$$\frac{\nu(x) \geq 3 \quad \nu'(x) = \nu(x) \quad (\nu'(x) \geq 0)}{(\ell_1, \nu) \xrightarrow{a} (\ell_2, \nu')}\ \mathtt{Rule}^{\ell_1 \to \ell_2}_{\mathtt{discrete}}$$

$$\frac{\nu'(x) = 0 \quad (\nu'(x) \leq 4)}{(\ell_2, \nu) \xrightarrow{a} (\ell_1, \nu')}\ \mathtt{Rule}^{\ell_2 \to \ell_1}_{\mathtt{discrete}}$$

*The antecedents in parenthesis are implied by the other antecedents and are thus not needed. Since the only variable $x$ is in the control variable sets of both locations, the $\tau$-transitions do not allow any state change:*

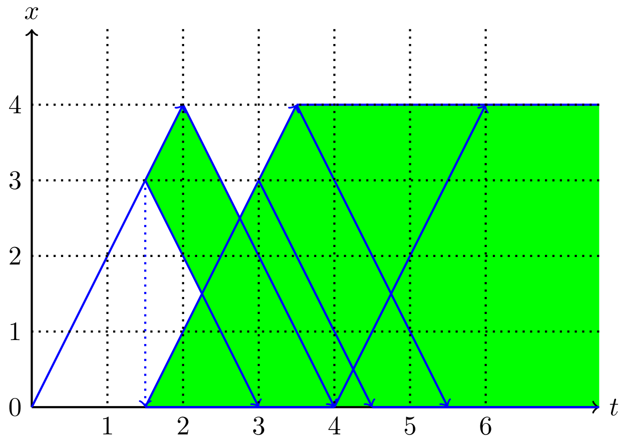
$$\frac{l \in Loc}{(l, \nu) \xrightarrow{\tau} (l, \nu)}\ \mathtt{Rule}^{\tau}_{\mathtt{discrete}}$$

*For the time steps we have the following rule instances:*

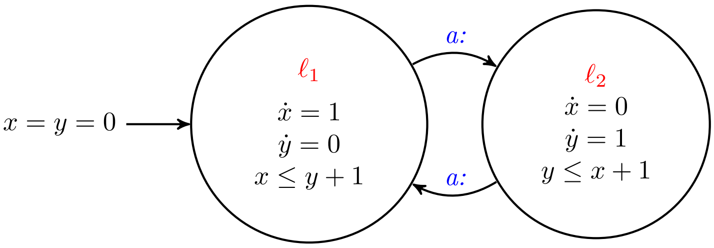
$$\frac{\nu'(x) \leq 4 \quad t \geq 0 \quad \nu'(x) = \nu(x) + 2t}{(\ell_1, \nu) \xrightarrow{a} (\ell_1, \nu')}\ \mathtt{Rule}^{\ell_1}_{\mathtt{time}}$$

$$\frac{\nu'(x) \geq 0 \quad t \geq 0 \quad \nu'(x) = \nu(x) - 2t}{(\ell_2, \nu) \xrightarrow{a} (\ell_2, \nu')}\ \mathtt{Rule}^{\ell_2}_{\mathtt{time}}$$

*The following picture visualizes the behavior of the system by depicting the possible values for $x$ at each point in time:*



**Example 8.2.** *Assume another hybrid automaton:*



*The formal definition is as follows:*

- $Loc = \{\ell_1, \ell_2\}$,
- $Var = \{x, y\}$,
- $Con(\ell_1) = Con(\ell_2) = \{x, y\}$,
- $Lab = \{\tau, a\}$,
- $Edge =$

$$\begin{array}{ll}\{ & (\ell_1,\ a,\ \{(\nu,\nu') \in V^2 \mid \nu = \nu'\},\ \ell_2),\\ & (\ell_2,\ a,\ \{(\nu,\nu') \in V^2 \mid \nu = \nu'\},\ \ell_1),\\ & (\ell_1,\ \tau,\ \{(\nu,\nu') \in V^2 \mid \nu = \nu'\},\ \ell_1),\\ & (\ell_2,\ \tau,\ \{(\nu,\nu') \in V^2 \mid \nu = \nu'\},\ \ell_2)\ \},\end{array}$$

- $Act(\ell_1) = \{f : \mathbb{R}_{\geq 0} \to V \mid \exists c_x, c_y \in \mathbb{R}. \forall t \in \mathbb{R}_{\geq 0}. f(t)(x) = t + c_x \wedge f(t)(y) = c_y\}$,
  $Act(\ell_2) = \{f : \mathbb{R}_{\geq 0} \to V \mid \exists c_x, c_y \in \mathbb{R}. \forall t \in \mathbb{R}_{\geq 0}. f(t)(x) = c_x \wedge f(t)(y) = t + c_y\}$,
- $Inv(\ell_1) = \{\nu \in V \mid \nu(x) \leq \nu(y) + 1\}$,
  $Inv(\ell_2) = \{\nu \in V \mid \nu(y) \leq \nu(x) + 1\}$,
- $Init = \{(\ell_1, \nu) \in \Sigma \mid \nu(x) = 0 \wedge \nu(y) = 0\}$.

*The behaviour can be visualized as follows by depicting the reachable $(x, y)$ value pairs (without representing the time):*

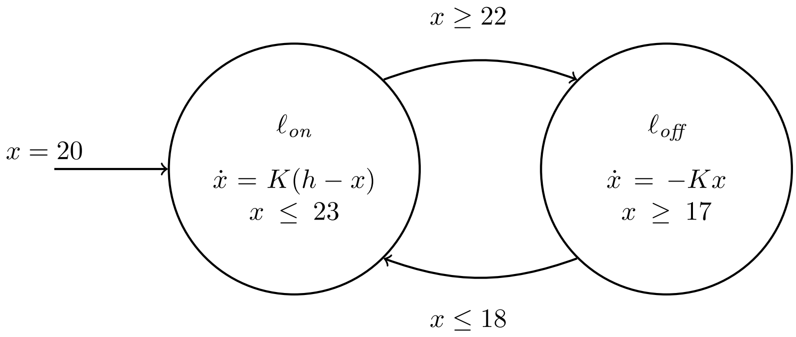
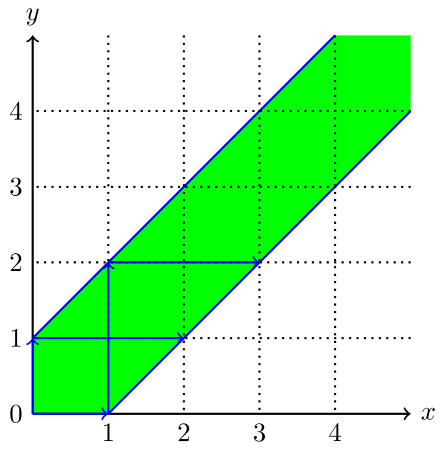


Figure 8.1: The hybrid automaton model of the thermostat



**Example 8.3 (Thermostat).** *Assume again the thermostat from Example 2.1. The modeling hybrid automaton is depicted on Figure 8.1.*

*In location $\ell_{on}$ the heater is on and the temperature raises according to the differential equation $\dot{x} = K(h - x)$. The location's invariant $x \leq 23$ assures that the heater turns off at latest when the temperature reaches $23°C$. Analogously for the location $\ell_{off}$, where the heater is off.*

*Control may move from location $\ell_{on}$ to $\ell_{off}$, switching the heater off, if the temperature is at least $22°C$, and from $\ell_{off}$ to $\ell_{on}$ if the temperature is at most $18°C$. The temperature $x$ does not change by jumping from $\ell_{on}$ to $\ell_{off}$ or from $\ell_{off}$ to $\ell_{on}$. Initially, the heater is on and the temperature is $20°C$.*

*Note that this model is non-deterministic. E.g., in location $\ell_{on}$, if the temperature is between $22°C$ and $23°C$, both time progress and switching the heater off are possible.*

**Example 8.4 (Water-level monitor).** *The hybrid automaton model for the water-level monitor Example 2.2 is depicted in Figure 8.2.*

*The automaton has two locations representing the control modes for refilling the first tank in $\ell_1$ or refilling the second tank in $\ell_2$. The water levels in the tanks are represented by the variables $x_1$ and $x_2$, being initially larger than $r_1$ resp. $r_2$ height units, i.e., initially $x_1 > r_1 \wedge x_2 > r_2$ holds.*

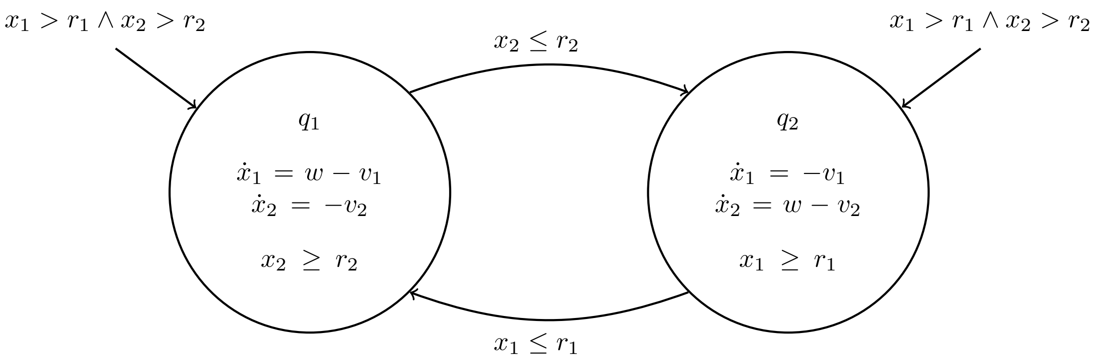


Figure 8.2: The hybrid automaton model of the water-level monitor

*Both tanks are leaking; the first tank looses $v_1$ height unit per time unit by leaking, the second tank $v_2$. When refilling a tank, $w$ height unit per time unit is refilled. That means, the activities in $\ell_1$ are represented by the differential equations $\dot{x}_1 = w - v_1$ and $\dot{x}_2 = -v_2$, and analogously for $\ell_2$. In order to increase the water level when refilling a tank we assume $w > v_1$ and $w > v_2$.*

*The invariant $x_2 \geq r_2$ of $\ell_1$ assures that the first tank is getting refilled only as long as there is enough water in the second tank (water level at least $r_2$). The hose will switch to refilling the second tank when the water level $x_2$ reaches $r_2$. This is done by taking the discrete transition from $\ell_1$ to $\ell_2$. Note that the transition's condition allows to switch only if $x_2$ is at most $r_2$, and the invariant assures that $x_2$ is at least $r_2$, such that the transition will be taken by the exact value $r_2$ of $x_2$. Refilling the second tank works analogously.*

*Note also that the discrete transitions can be taken only if the target location's invariant $x_1 \geq r_1$ is not violated. It can be shown that both invariants are globally valid, and thus the discrete transitions are never blocked by the invariants.*

**Example 8.5 (Bouncing Ball).** *The hybrid automaton model of the bouncing ball from Example 2.3 is depicted on Figure 8.3. Initially the height of the ball $x_1$ is larger or equal 0 (height 0 corresponds to the earth and positive height above the earth) and its speed $x_2$ is positive, stating that the ball is initially raising.*

*The automaton has a single location $\ell_0$. Time progress in this location corresponds to the raising and falling of the ball. The differential equation $\dot{x}_1 = x_2$ defines $x_2$ as the derivative of the height, i.e., the ball's speed, and $\dot{x}_2 = -g$ with $g$ the gravity constant defines the speed change due to gravity.*

*The ball can raise and fall only as long as it has a non-negative height as stated by the invariant $x_1 \geq 0$. After raising and reaching the highest point, it starts falling and reaches the earth when $x_1 = 0$ and $x_2 < 0$. Then it bounces, represented by the single discrete transition. Note that the bounce is forced by the invariant. The bounce changes the speed's direction and reduces its absolute value due to some loss of kinetic energy during bouncing as denoted by $x_2 := -cx_2$. After bouncing, $x_1$ is still 0 but $x_2$ is now positive, and the ball raises again.*

For the ease of modeling, also hybrid systems can be modeled componentwise. The resulting global system is given by the parallel composition of the different components. The parallel composition of hybrid automata extends the definition of the parallel composition for LTSs as follows.

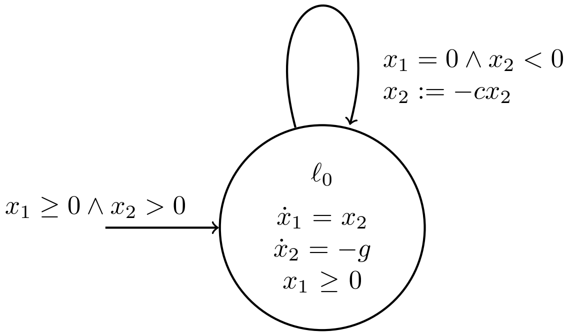


Figure 8.3: The hybrid automaton model of the bouncing ball

**Definition 8.3 (Parallel composition of hybrid automata).** *Let*

$$\begin{aligned}\mathcal{H}_1 &= (Loc_1, Var, Con_1, Lab_1, Edge_1, Act_1, Inv_1, Init_1) \text{ and}\\ \mathcal{H}_2 &= (Loc_2, Var, Con_2, Lab_2, Edge_2, Act_2, Inv_2, Init_2)\end{aligned}$$

*be two hybrid automata. The* parallel composition *or* product $\mathcal{H}_1||\mathcal{H}_2$ *of* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *is defined to be the hybrid automaton*

$\mathcal{H}_1||\mathcal{H}_2$

$$\mathcal{H} = (Loc, Var, Con, Lab, Edge, Act, Inv, Init)$$

*with*

- *The LTS part* $(Loc, Var, Con, Lab, Edge, Init)$ *equals the parallel composition*

  $$(Loc_1, Var, Con_1, Lab_1, Edge_1, Init_1)||(Loc_2, Var, Con_2, Lab_2, Edge_2, Init_2)$$

  *of the LTS parts of the components,*
- $Act(\ell_1, \ell_2) = Act_1(\ell_1) \cap Act_2(\ell_2)$ *for all* $(\ell_1, \ell_2) \in Loc$*, and*
- $Inv(\ell_1, \ell_2) = Inv_1(\ell_1) \cap Inv_2(\ell_2)$ *for all* $(\ell_1, \ell_2) \in Loc$*.*

## 8.2 Approximative State Set Representations

The reachability problem for hybrid automata is in general undecidable. Nevertheless, incomplete algorithms exist for reachability analysis, which allow to check safety properties of the systems. Most algorithms compute *over-approximations* of the reachability relation. Generally there are two kinds of approaches:

1. We can build a finite abstraction of the state space, and compute reachability for the abstract system. We have already seen a typical example for this approach: the region automata construction for timed automata. As this abstraction is based on bisimulation, it is not over-approximating. Another example is the on-the-fly refinement of the predicate abstraction during the fixed-point computation of the last chapter. This abstraction is in general over-approximating. Spurious counterexamples can be eliminated by (counterexample-guided) abstraction refinement (realized by splitting the abstract states).

```
Input: Set Init of initial states.
Algorithm:
    $R^{\mathsf{new}} := \mathsf{Init}$;
    $R := \emptyset$;
    while ($R^{\mathsf{new}} \neq \emptyset$){
        $R$        $:= R \cup R^{\mathsf{new}}$;
        $R^{\mathsf{new}}$   $:= \mathsf{Reach}(R^{\mathsf{new}}) \backslash R$;
    }
Output: Set $R$ of reachable states.
```

Figure 8.4: General reachability computation algorithm

2. An alternative approach is to compute reachability for the original system, without abstraction, but over-approximating the set of reachable states.

In this chapter we deal with the latter approach. The general forward reachability computation procedure can be specified by the algorithm depicted on Figure 8.4. Note that if the state space is infinite, then this algorithm does not always terminate.

In order to implement the above algorithm, we must solve two problems:

1. We must be able to store the current reachable sets $R$ and $R^{\mathsf{new}}$, and build their union, intersection, etc.

2. We must be able to compute $\mathsf{Reach}(P)$ for a set $P$.

Note that in general the reachable sets cannot be represented exactly, as for general hybrid automata even the reachability relation due to a single time step can be undecidable. We solve the above problems by

1. over-approximating the reachable sets by sets having certain geometric forms, such that the needed operations can be done efficiently, and

2. over-approximate $\mathsf{Reach}(P)$ in each step in the above procedure.

In this chapter we discuss the first point, and introduce representations of state sets. Afterwards, in the next chapter we discuss methods for the over-approximation of $\mathsf{Reach}(P)$. Putting the two together, we can implement the algorithm on Figure 8.4. To get an intuition, an example reachability approximation using the reachability algorithm for some hybrid automaton is visualized on Figure 8.5. The exact continuous behavior is depicted on the left, while the approximation is drawn on the right.

The *geometry* chosen to represent reachable sets has a crucial effect on the practicability of the whole procedure. Usually, the more complex the geometry,

1. the more costly is the *storage* of the sets,

2. the more difficult it is to *perform operations* like union and intersection, and

3. the more elaborate is the *computation of new reachable* sets, but

4. the better the *approximation* of the set of reachable states.

Choosing the geometry has to be a *compromise* between these impacts. The representation should allow *efficient computation* of the operations for

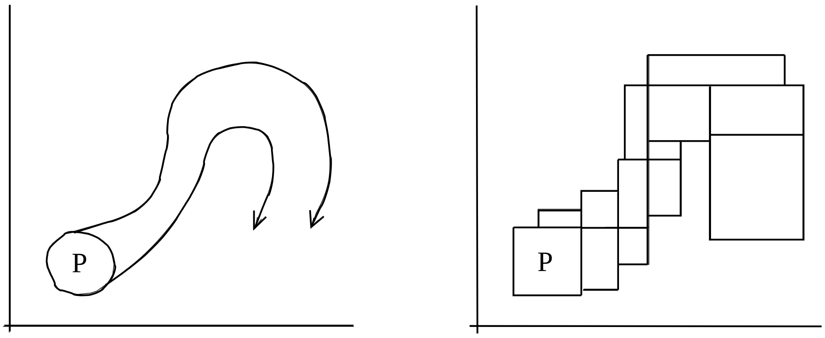


Figure 8.5: An example reachability approximation for a hybrid automaton

- membership relation,
- union,
- intersection,
- subtraction,
- test for emptiness.

In the remaining part of this chapter we have a closer look at representation by

- orthogonal polyhedra in Section 8.3 and
- convex polyhedra in Section 8.4.

# Exercises

**Exercise 8.1.** Show that the parallel composition of hybrid automata is commutative and associative.

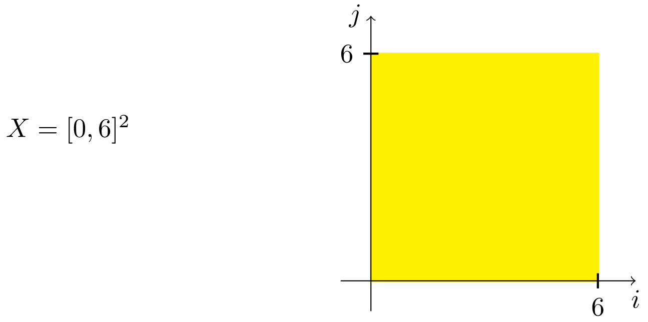


Figure 8.6: An example domain

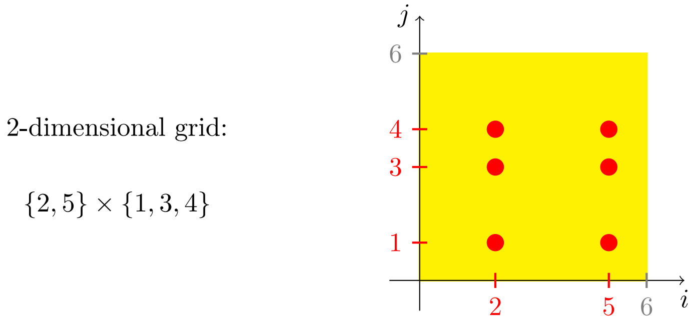


Figure 8.7: An example grid

## 8.3 Orthogonal Polyhedra

The content of this section is based on the publications [BMP99] and [SK03].

### 8.3.1 Definition

$X$ As state space *domain* we consider a bounded subset $X = [0,m]^d \subseteq \mathbb{R}^d$ $(m \in \mathbb{N}_{>0})$ of the reals (can be extended to $X = \mathbb{R}^d_+$). Elements of $X$ are denoted by $x = (x_1, \ldots, x_d)$, the zero vector by $\mathbf{0}$, and the unit vector by $\mathbf{1}$. For our running example we define the domain $X = [0,6]^2$, depicted on Figure 8.6.

**Definition 8.4.** *A $d$-dimensional* grid *associated with the domain $X = [0,m]^d \subseteq \mathbb{R}^d$ $(m \in \mathbb{N}_{>0})$ is a product of $d$ subsets of $\{0,1,\ldots,m-1\}$.*

An example grid is depicted on Figure 8.7.

$\mathbf{G}$ **Definition 8.5.** *The* elementary grid *associated with $X = [0,m]^d \subseteq \mathbb{R}^d$ $(m \in \mathbb{N}_{>0})$ is $\mathbf{G} = \{0,1,\ldots,m-1\}^d \subseteq \mathbb{N}^d$.*

An example elementary grid is depicted on Figure 8.8 The grid admits a natural *partial order* with $(m-1,\ldots,m-1)$ on the top and $\mathbf{0}$ as bottom, as show on Figure 8.9.

The set of subsets of the elementary grid $\mathbf{G}$ forms a
*Boolean algebra* $(2^{\mathbf{G}}, \cap, \cup, \sim)$ under the set-theoretic operations

$G = \{0, \ldots, 5\} \times \{0, \ldots, 5\}$

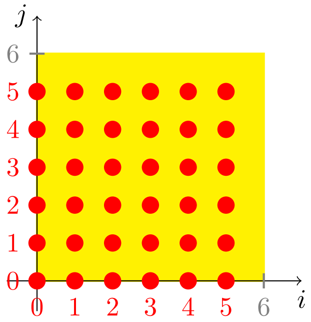


Figure 8.8: An example elementary grid

$G = \{0, \ldots, 5\} \times \{0, \ldots, 5\}$

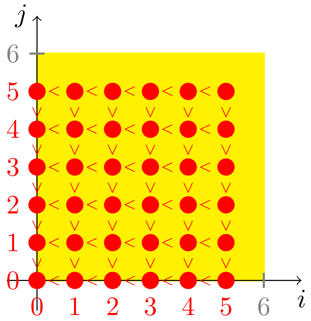


Figure 8.9: The partial order on the grid points of an elementary grid

- $A \cup B$
- $A \cap B$
- $\sim A = \mathsf{G} \backslash A$

for $A, B \subseteq \mathbf{G} \subset \mathbb{N}^d$. An example cut of two subsets is shown on Figure 8.10.

**Definition 8.6 (Elementary box).** *The* elementary box *associated with a grid point* $x = (x_1, \ldots, x_d)$ *is* $B(x) = [x_1, x_1 + 1] \times \ldots, \times [x_d, x_d + 1]$. *The set of elementary boxes is denoted by* $\mathbf{B}$.

$B, \mathbf{B}$

An example of an elementary box of a grid point is shown on Figure 8.11.

**Definition 8.7 (Orthogonal polyhedra).** *An* orthogonal polyhedron $P$ *is a union of elementary boxes, i.e., an element of* $2^{\mathbf{B}}$.

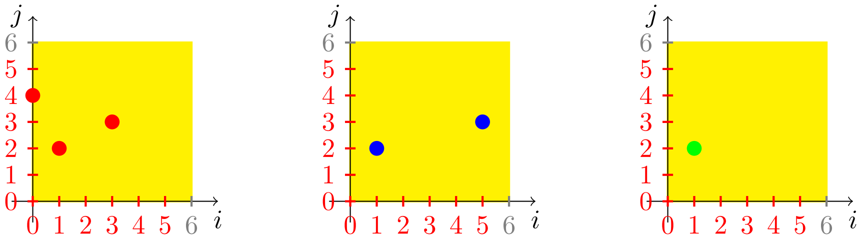
An example of an orthogonal polyhedron is shown on Figure 8.12.

The set of orthogonal polyhedra forms a *Boolean algebra* $(2^{\mathbf{B}}, \sqcap, \sqcup, \neg)$ with the operations

- $A \sqcup B = A \cup B$
- $A \sqcap B = cl(int(A) \cap int(B))$
- $\neg A = cl(X \backslash A)$

where

- *int* is the interior operator yielding the largest open set $int(A)$ contained in $A$, and



$$\{(0,4),(1,2),(3,3)\} \quad \cap \quad \{(1,2),(5,3)\} \quad = \quad \{(1,2)\}$$

Figure 8.10: An example cut of two subsets of an elementary grid

$$B((2,4)) = [2,3] \times [4,5]$$

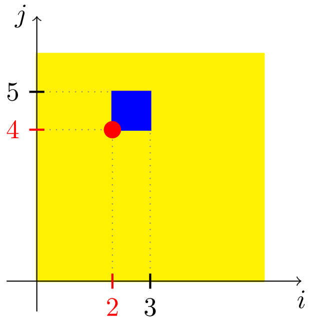


Figure 8.11: The elementary box of a grid point

$$\{B((2,4))\} \cup \{B((3,4))\} \cup$$
$$\{B((2,3))\} \cup \{B((3,3))\} \cup$$
$$\{B((2,2))\} \cup$$
$$\{B((2,1))\}$$

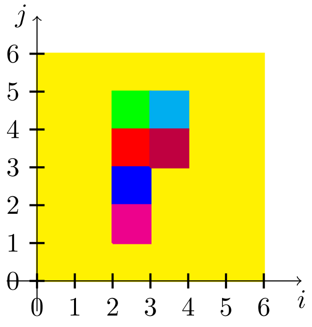


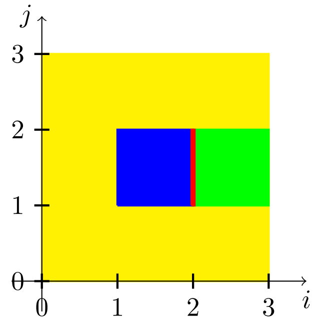
Figure 8.12: An example orthogonal polyhedron

$$A \sqcap B = cl(int(A) \cap int(B))$$

$([1,2] \times [1,2]) \sqcap ([2,3] \times [1,2]) =$

$cl(((1,2) \times (1,2)) \sqcap ((2,3) \times (1,2))) =$

$cl(\emptyset) = \emptyset$



Note: $([1,2] \times [1,2]) \cap ([2,3] \times [1,2]) = [2,2] \times [1,2]$

Figure 8.13: An example for the cut of two orthogonal polyhedra

- $cl$ is the topological closure operator yielding the smallest closed set $cl(A)$ containing $A$.

For the operations $\sqcap$ and $\neg A$ we need the interior and closure operations in order to be closed under orthogonal polyhedra, i.e., for the result being an orthogonal polyhedra. Intuitively, we use the standard set operations but want to compute the cut and negation not in terms of points but in terms of elementary boxes.

Figure 8.13 gives an example for $\sqcap$. Without the interior operation the result would be the line between the two elementary boxes. With the interior operation the result is the empty set.

Figure 8.14 gives an example for the negation. Without building the closure of the set negation the result would be an open set.

The *bijection between* $\mathbf{G}$ *and* $\mathbf{B}$ which associates every elementary box with its leftmost corner generates an *isomorphism between* $(2^{\mathbf{G}}, \cap, \cup, \sim)$ *and* $(2^{\mathbf{B}}, \sqcap, \sqcup, \neg)$. Thus we can switch between point-based and box-based terminology according to what serves better the intuition, as illustrated on Figure 8.15.

### 8.3.2 Representation

We need an representation of orthogonal polyhedra that allows efficient computation. The representations we consider are based on the vertices of orthogonal polyhedra. Below we give some definitions we need for the definition of a vertex.

**Definition 8.8 (Color function).** *Let $P$ be an orthogonal polyhedron. The* color function *$c : X \to \{0,1\}$ is defined by*

$$c(x) = \begin{cases} 1 & \textit{if } x \textit{ is a grid point and } B(x) \subseteq P \\ 0 & \textit{otherwise} \end{cases}$$

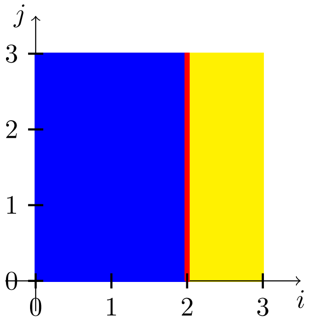
*for all $x \in X$.*
*If $c(x) = 1$ we say that $x$ is* black *and that $B(x)$ is* full.
*If $c(x) = 0$ we say that $x$ is* white *and that $B(x)$ is* empty.

$$\neg A = cl(\sim A)$$

$\neg([0,2] \times [0,3]) =$

$cl(\sim ([0,2] \times [0,3])) =$

$cl((2,3] \times [0,3])) = [2,3] \times [0,3]$



Note: $\sim ([0,2] \times [0,3]) = (2,3] \times [0,3]$

Figure 8.14: An example for the negation of an orthogonal polyhedron

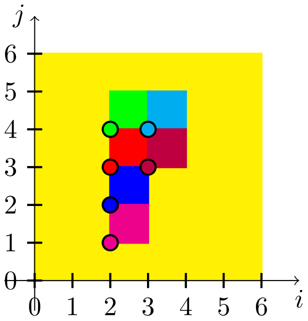


Figure 8.15: Bijection between **G** and **B**

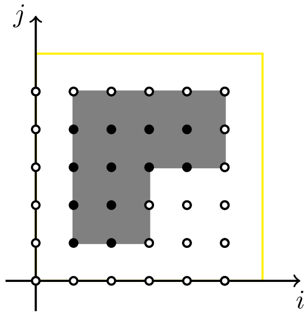


Figure 8.16: The coloring function for an orthogonal polyhedron

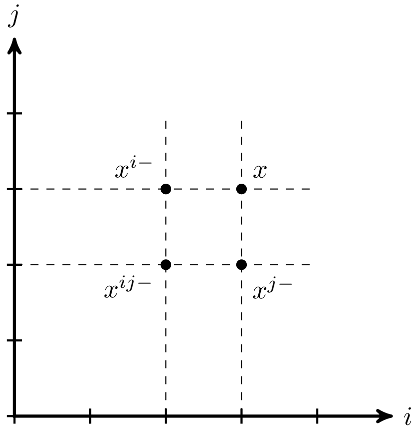


Figure 8.17: The predecessors of a grid point

Note that $c$ almost coincides with the characteristic function of $P$ as a subset of $X$. It differs from it only on right-boundary points. The coloring for an example orthogonal polyhedron is shown in Figure 8.16.

The following definitions capture the intuitive meaning of a facet and a vertex and, in particular, that the boundary of an orthogonal polyhedron is the union of its facets.

**Definition 8.9 ($i$-predecessor).** *The* $i$-predecessor *of a grid point* $x = (x_1, \ldots, x_d) \in X$ *is* $x^{i-} = (x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_d)$. *We use* $x^{ij-}$ *to denote* $(x^{i-})^{j-}$. *When* $x$ *has no* $i$*-predecessor, we write* $\perp$ *for the predecessor value.*

The above definition is illustrated in Figure 8.17.

**Definition 8.10 (Neighborhood).** *The* neighborhood *of a grid point* $x$ *is the set*

$$\mathcal{N}(x) = \{x_1 - 1, x_1\} \times \ldots \times \{x_d - 1, x_d\}$$

*(the vertices of a box lying between* $x - \mathbf{1}$ *and* $x$*). For every* $i$*,* $\mathcal{N}(x)$ *can be partitioned into left and right* $i$*-neighborhoods*

$$\mathcal{N}^{i-}(x) = \{x_1 - 1, x_1\} \times \ldots \times \{x_i - 1\} \times \{x_d - 1, x_d\}$$

*and*

$$\mathcal{N}^i(x) = \{x_1 - 1, x_1\} \times \ldots \times \{x_i\} \times \{x_d - 1, x_d\}.$$

**Definition 8.11 ($i$-hyperplane).** *An $i$-*hyperplane *is a $(d-1)$-dimensional subset $H_{i,z}$ of $X$ consisting of all points $x$ satisfying $x_i = z$.*

The above definition is illustrated in Figure 8.18.

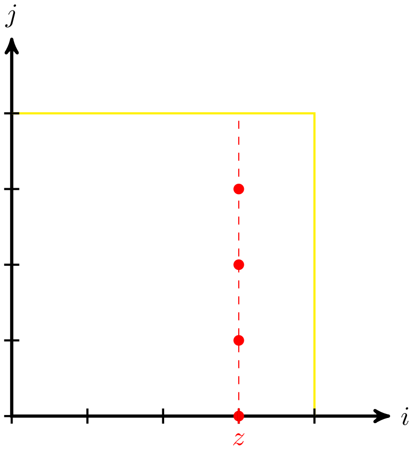


Figure 8.18: An $i$-hyperplane of $X$

Note that the facets of orthogonal polyhedra are $d-1$-dimensional polyhedra, and as such, they are subsets of $i$-hyperplanes. For the definition of facets we make use of the fact that the coloring changes on facets. Thereby we need to pay attention to white vertices lying on the boundaries to the “right”. For example, the orthogonal polyhedron in Figure 8.16 has white grid points on the top and at the right that belong to facets or are vertices.

**Definition 8.12 ($i$-facet).** *An $i$-*facet *of an orthogonal polyhedron $P$ with color function $c$ is*

$$F_{i,z}(P) = cl\{x \in H_{i,z} | c(x) \neq c(x^{i-})\}$$

*for some integer $z \in [0, m)$.*

**Definition 8.13 (Vertex).** *A* vertex *is a non-empty intersection of $d$ distinct facets. The set of vertices of an orthogonal polyhedron $P$ is denoted by $V(P)$.*

In Figure 8.19 the vertices of an orthogonal polyhedron are marked red.

**Definition 8.14 ($i$-vertex-predecessor).** *An $i$-*vertex-predecessor *of a grid point $x = (x_1, \ldots, x_d) \in X$ is a vertex of the form $(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_d)$ for some integer $z \in [0, x_i]$. When $x$ has no $i$-vertex-predecessor, we write $\perp$ for its value.*
*The* first $i$-vertex-predecessor *of $x$, denoted by $x^{i\leftarrow}$, is the one with the maximal $z$.*

Figure 8.20 shows the first $i$-vertex-predecessors of some points for our example orthogonal polyhedron.

A *representation scheme* for $2^{\mathbf{B}}$ ($2^{\mathbf{G}}$) is a set $\mathcal{E}$ of syntactic objects such that there is a surjective function $\phi$ from $\mathcal{E}$ to $2^{\mathbf{B}}$, i.e., every syntactic object represents at most one polyhedron

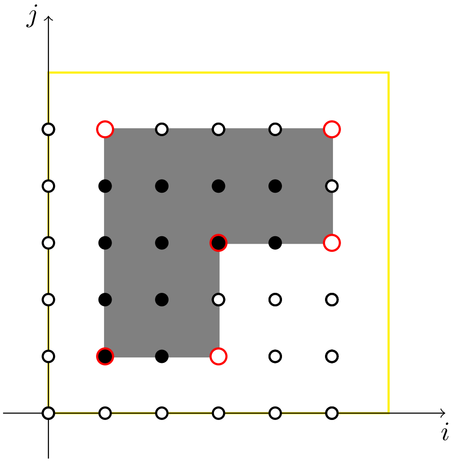


Figure 8.19: The vertices of an orthogonal polyhedron

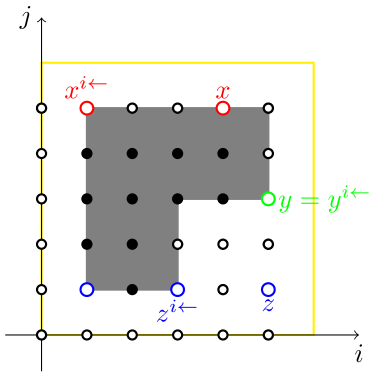


Figure 8.20: The first $i$-vertex-predecessors of some points and an orthogonal polyhedron

and every polyhedron has at least one corresponding object. If $\phi$ is an injection we say that the representation is *canonical*, i.e., every polyhedron has a unique representation.

There are different representation schemes we could consider. A naive way would be an *explicit representation* consisting of the enumeration of the color values on every grid point, i.e., a $d$-dimensional zero-one array with $m^d$ entities. However, such a representation would be both for storage and computation very inefficient.

Another possibility would be a *Boolean representation*, specifying an orthogonal polyhedron as a Boolean combination of inequalities of the form $x_i \geq z$. Note that this representation is non-canonical.

In this section we consider the following vertex-based representations:

- *Vertex representation*: An orthogonal polyhedron $P$ is represented by the set $\{(x, c(x)) \mid x \textit{ is a vertex of } P\}$, i.e., the vertices of $P$ along with their color. This representation is canonical.
  Note that the vertices alone would not yield a unique representation scheme. Figure 8.21 shows two orthogonal polyhedra having the same vertex sets but assigning different colors to the vertices.
  Note also that not every set of points and colors is a valid representation of a polyhedron.

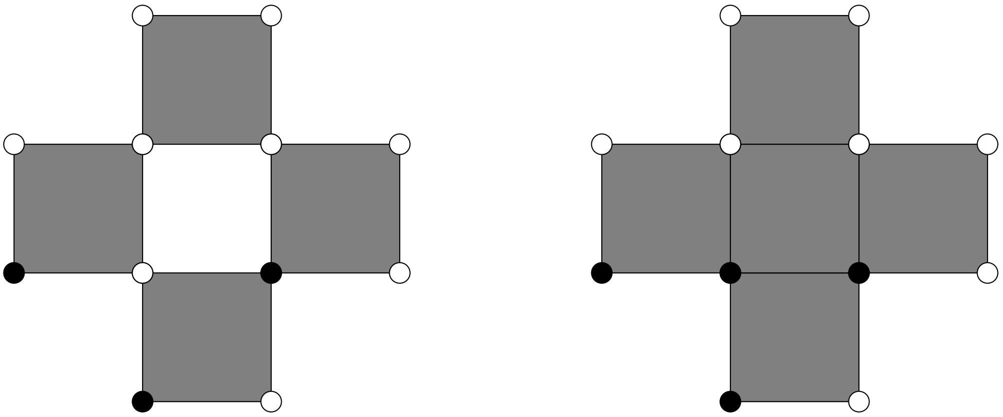


Figure 8.21: Two vertex representations agreeing on the vertices but having different vertex colors

- *Neighborhood representation*: An orthogonal polyhedron is represented by its vertices along with the colors of all the $2^d$ points in the neighborhoods of the vertices.
- *Extreme vertex representation*: This representation keeps the vertices and the *parity* of the number of black points in the neighborhood of the vertices. In fact, it suffices to keep only vertices with odd parity.

Why we choose the above representations, especially in the case of the neighborhood and the extreme vertex representations, will become clear in the following when we define the algorithms for the required operations on orthogonal polyhedra. We do not discuss all needed operations. Instead, we describe the algorithms for the membership problem and for the computation of the intersection of two orthogonal polyhedra, only.

### 8.3.3 Membership Problem

Next we discuss how to solve the *membership problem* for orthogonal polyhedra, based on the three different representation schemes suggested above. Given a representation of a polyhedron $P$ and a grid point $x$, the membership problem is the problem to determine $c(x)$, that is, whether $B(x) \subseteq P$.

#### Membership Problem for the Vertex Representation

We make use of the following observations:

- A point $x$ is on an $i$-facet iff
$$\exists x' \in \mathcal{N}^i(x).\ c(x'^{i-}) \neq c(x').$$
- A point $x$ is a vertex iff
$$\forall i \in \{1, \ldots, d\}.\ \exists x' \in \mathcal{N}^i(x).\ c(x'^{i-}) \neq c(x').$$
- A point $x$ is not a vertex iff
$$\exists i \in \{1, \ldots, d\}.\ \forall x' \in \mathcal{N}^i(x).\ c(x'^{i-}) = c(x').$$

Given an orthogonal polyhedron in vertex representation, we can compute the color of a non-vertex grid point recursively, using the following lemma:

**Lemma 8.1 (Color of a non-vertex).** *Let $x$ be a non-vertex. Then there exists a direction $j \in \{1, \dots, d\}$ such that*

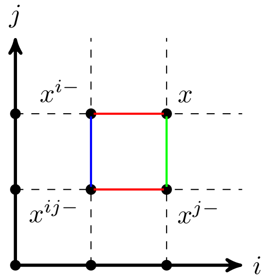
$$\forall x' \in \mathcal{N}^j(x)\backslash\{x\}.\ c(x'^{j-}) = c(x'). \tag{8.1}$$

*Let $j$ be such a direction. Then $c(x) = c(x^{j-})$.*

*Proof.* A point $x$ is not a vertex iff there is a dimension in that the point is not on a facet, i.e.,

$$\exists i \in \{1, \dots, d\}.\ \forall x' \in \mathcal{N}^i(x).\ c(x'^{i-}) = c(x'). \tag{8.2}$$

Thus $j$ always exists. Assume a $j$ satisfying 8.1 and an $i$ satisfying 8.2. If $i = j$, then the case is straightforward. Otherwise, if $i \neq j$, for $i$ we have $c(x^{i-}) = c(x)$ and $c(x^{ij-}) = c(x^{j-})$. For $j$ we have $c(x^{ij-}) = c(x^{i-})$. By the transitivity of "=" we get $c(x) = c(x^{j-})$.



□

□

Consequently we can calculate the color of a non-vertex $x$ based on the color of all points in $\mathcal{N}(x) - \{x\}$: just find some $j$ satisfying the conditions of the above lemma and let $c(x) = c(x^{j-})$.

**Theorem 8.1.** *For a domain $X = [0, n]^d$, the membership problem for vertex representation can be solved in time $\mathcal{O}(n^d d 2^d)$ using space $\mathcal{O}(n^d)$.*

*Proof.* We start at $x$ and recursively determine the membership of all the $2^d - 1$ points in $\mathcal{N}(x)\backslash\{x\}$. Termination of the recursion is guaranteed because we go down in the partial order on $2^{\mathsf{G}}$ and either encounter vertices or reach the origin. We must recursively determine the color of at most $n^d$ grid points. For each of them we must check at most $d$ dimensions if they satisfy the condition of the lemma on the color of a non-vertex. Checking the condition in a dimension invokes $2^{d-1} - 1$ color comparisons. □ □

However, this algorithm is not very efficient, because in the worst-case one has to calculate the color of all the grid points between $\mathbf{0}$ and $x$. We can improve it using the notion of an *induced grid*: let the *i-scale* of $P$ be the set of the $i$-coordinates of the vertices of $P$, and let the induced grid be the Cartesian product of its $i$-scales. The induced grid is the smallest (coarsest) grid containing all the vertices. Every rectangle in the induced grid has a uniform color. Calculating the color of a point reduces to finding its closest "dominating" point on the induced grid and applying the algorithm to that grid in $\mathcal{O}(n^d d 2^d)$ time, where $n$ is the number of grid points in the induced grid. The approach is illustrated in Figure 8.22.

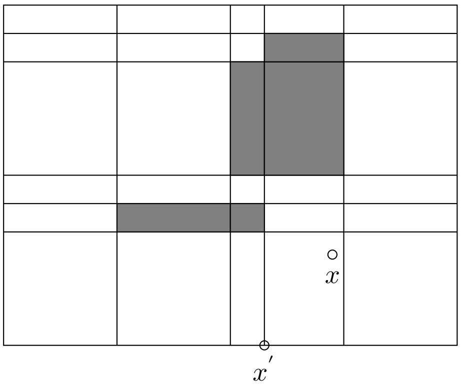


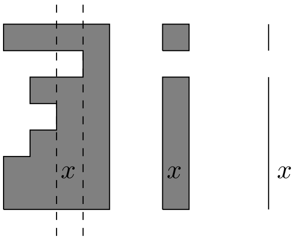
Figure 8.22: Membership problem on an induced grid

### Membership Problem for the Neighborhood Representation

The solution of the membership problem for the neighborhood representation is based on projection.

**Definition 8.15 ($i$-slice and $i$-section).** *Let $P$ be an orthogonal polyhedron and $z$ an integer in $[0, m)$.*

- *The* $i$-slice of $P$ at $z$ *is the $d$-dimensional orthogonal polyhedron $J_{i,z}(P) = P \sqcap \{x | z \leq x_i \leq z + 1\}$.*
- *The* $i$-section of $P$ at $z$ *is the $(d-1)$-dimensional orthogonal polyhedron $\mathcal{J}_{i,z}(P) = J_{i,z}(P) \cap H_{i,z}$.*



The membership of $x = (x_1, \ldots, x_d)$ can be reduced to membership in $\mathcal{J}_{i,x_i}(P)$, which is a $(d-1)$-dimensional problem: a $x$ is contained in an orthogonal polyhedron $P$ iff it is contained in the $i$-section of $P$ at $x_i$. By successively reducing dimensionality for every $i$ we obtain a point whose color is that of $x$.

For the computation of an $i$-section of an orthogonal polyhedron $P$ we first observe that the vertices of the $i$-section are points $x$ in the corresponding $i$-hyperplane that lie on a facet of $P$ in each dimension $j \neq i$. I.e., a point $x$ in the corresponding $i$-hyperplane is a vertex of the $i$-section iff (1) $x$ has a first $i$-vertex-predecessor $y = x^{i\leftarrow} \neq \perp$ such that (2) for each dimension $j \neq i$ there is a $j$-facet to the right of $y$ in dimension $i$. The second condition assures that this facet intersects with the $i$-hyperplane.

The neighborhood representation provides us with the vertices and the colors of their neighbors. This coloring information can be used to determine in which directions (relative to the vertex) the faces defining the vertex lie and thus to check the second condition.

Let us take as an example the two-dimensional case. Figure 8.23 lists all 16 possible neighborhood colorings of a grid point $x$ for $d = 2$. The colorings in the first row define non-vertex grid points, whereas the second and third rows contain vertices. Let $i$ be the horizontal and $j$ the vertical dimension. For the vertices, i.e., the points in the second and third rows, we can observe that there is a $j$-facet to the right of $x$ (i.e., to the right in dimension $i$) iff the color of $x$ differs from the color of $x^{j-}$, i.e., the grid point below $x$ (the point to the left from $x$ in dimension $j$). This is the case for all grid points in the third row, whereas the grid points of the second row do not fulfill this condition.

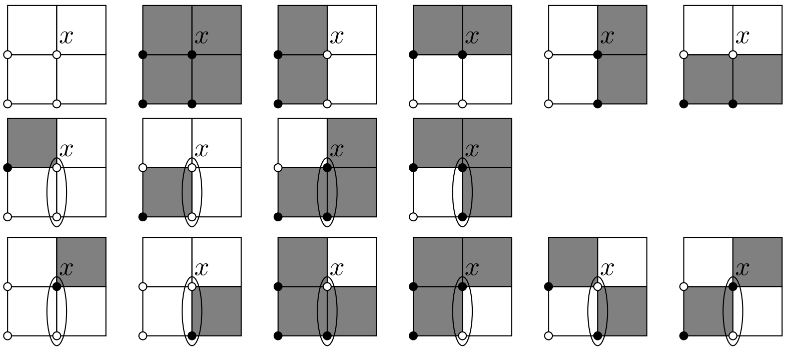


Figure 8.23: Possible neighborhood colorings in the two-dimensional case

We can use this information to define a projection operation. We introduce an $\mathcal{O}(n \log n)$ membership algorithm for the neighborhood representation, based on successive projections of $P$ into polyhedra of smaller dimension.

We use the following lemma to calculate the neighborhood representation of an $i$-section.

**Lemma 8.2 (Vertex of a section).** *Let $P$ be an orthogonal polyhedron and let $P'$ be its $i$-section at $x_i = z$. A point $x$ is a vertex of $P'$ iff $y = x^{i\leftarrow} \neq \perp$ and for every $j \neq i$ there exists $x' \in \mathcal{N}^i(y) \cap \mathcal{N}^j(y)$ such that $c(x'^{j-}) \neq c(x')$.*
*Moreover, when this condition is true, the neighborhood of $x$ relative to $\mathcal{J}_{i,z}(P)$ is given by $\mathcal{N}^i(y)$.*

*Proof.* Assume $x$ is a vertex of $P'$. Then there is a facet of $P$ orthogonal to $i$ that goes through $x$. This is possible only if there is a vertex $x^{i\leftarrow} = y = (x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_d)$ left to $x$ in the dimension $i$. The vertex $y$ lies on a facet in all dimensions $j \neq i$, implying the existence of an $x' \in \mathcal{N}^i(y) \cap \mathcal{N}^j(y)$ such that $c(x'^{j-}) \neq c(x')$. Finally, since the facet goes from $y$ to the right in $i$ through $x$, we have that $c(\mathcal{N}^i(y)) = c(\mathcal{N}^i(x))$.

Assume conversely $y = x^{i\leftarrow}$ exists and it satisfies the condition. Then $c(\mathcal{N}^i(x)) = c(\mathcal{N}^i(y))$, because otherwise, by the above reasoning, there would be a vertex between $x$ and $y$. Hence $x$ satisfies the condition. □

□

The resulting projection algorithm is illustrated on an example in Figure 8.24.

**Theorem 8.2 (Membership problem for the neighborhood representation).** *The membership problem for the neighborhood representation can be solved in time*
*$\mathcal{O}(nd^2(\log n + 2^d))$.*

*Proof.* For a $d$-dimensional orthogonal polyhedron $P$ with $n$ vertices we can determine those vertices $y$ which are $x^{i\leftarrow}$ for some $x \in H_{i,z}$ in $\mathcal{O}(nd \log n)$ steps.

There are most $n$ such points. With the above lemma we can determine in $\mathcal{O}(d2^d)$ time whether one such point is a vertex of the section. Hence it takes $\mathcal{O}(nd(\log n + 2^d))$ to get rid of

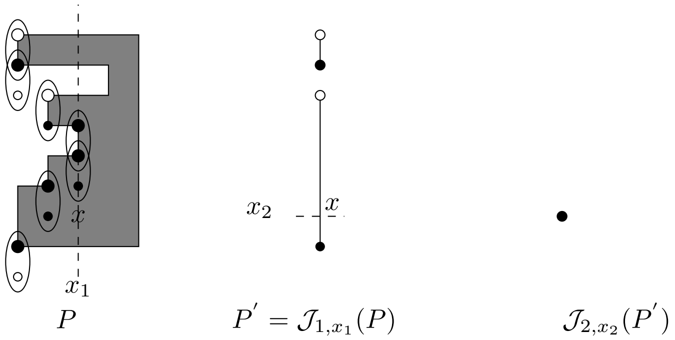

$x_1$
$x$
$x_2$
$x$
$P$
$P^{'} = \mathcal{J}_{1,x_1}(P)$
$\mathcal{J}_{2,x_2}(P^{'})$

Figure 8.24: Solving the membership problem by projection for the neighborhood representation

one dimension.

This is repeated $d$ times until $P$ is contracted into a point. □ □

A similar algorithm with the same complexity can be used to calculate the *color of all the points in a neighborhood of* $x$. The algorithm takes double slices ($d$-dimensional thick sections of width two) of $P$, and successively reduces $P$ into the neighborhood of $x$. This variation of the algorithm is used for doing Boolean operations.

### Membership Problem for the Extreme Vertex Representation

The *extreme vertex representation* can be viewed as a compaction of the neighborhood representation. Instead of maintaining all the neighborhood of each vertex, it suffices to keep only the *parity* of the number of black points in that neighborhood – in fact it suffices to keep only vertices with odd parity.

**Definition 8.16 (Extreme point).** *We use parity*($x$) *to denote the parity of the number of black points in the neighborhood* $\mathcal{N}(x)$ *of a grid point* $x$. *The grid point* $x$ *is said to be* extreme *if parity*$(x) = 1$.

**Lemma 8.3.** *An extreme point is a vertex.*

*Proof.* By induction on the dimension $d$. The base case $d = 1$ is immediate. For $d > 1$, choose $i \in \{1, \ldots, d\}$. Exactly one of $\mathcal{N}^{i-}(x)$ and $\mathcal{N}^{i}(x)$ contains an odd number of black points. Assume w.l.o.g. that it is $\mathcal{N}^{i}(x)$. By induction hypothesis $x$ is a vertex in $\mathcal{J}_{i,x_i}(P)$. I.e., for every $j \neq i$ there exists $x' \in \mathcal{N}^{j}(x)$ such that $c(x'^{j-}) \neq c(x')$. Since one cannot have $c(x') = c(x'^{i-})$ for all $x' \in \mathcal{N}^{i}(x)$, $x$ is a vertex of $P$. □ □

The converse is not true, i.e., vertices in general need not be extreme.

An *extreme vertex representation* consists in representing an orthogonal polyhedron by the set of its extreme vertices. (Additionally, the color of the origin is stored in a bit. From this information the colors of all extreme vertices can be inferred.)

Note that for $d = 1$ all vertices are extreme and hence the vertex and extreme vertex representations coincide. Figure 8.25 gives some two-dimensional representation examples, where the extreme vertices of the objects are marked red. For the objects in the left and in the middle all vertices are extreme. For the object in the right all 4 vertices in the middle are not extreme.

The *membership problem* is solved again by *projection.* To define a projection operation we need again a rule to determine which points of an $i$-section are extreme vertices of the projection.

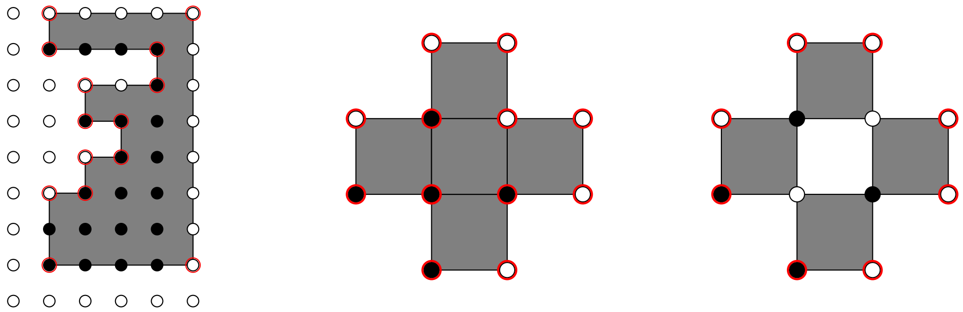


Figure 8.25: Examples for the extreme vertex representation

Let us again first give an intuition in the two-dimensional case for the role of the parity of a vertex in the projection. The last two lines in Figure 8.23 list all possible neighborhoods of vertices of a two-dimensional orthogonal polyhedron. All but the last two vertices are extreme. Note that these last two cases are the only ones representing vertices where two facets meet each other.

Intuitively, the basic idea for the solution of the membership problem for the extreme vertex representation is the following: We again use projection. Given a point $x$ on the $i$-hyperplane, we count its extreme $i$-vertex-predecessors to determine if $x$ is an extreme vertex of the $i$-section. If there is an even number $2n$ of such vertices, than we can conclude that $n$ facets started and ended at those vertices but there is no "open" facet to the left in $i$ and thus the considered point $x$ does not lie on any facet that goes orthogonal through the $i$-hyperplane. Consequently, $x$ is not a vertex of the $i$-section. Otherwise, if there is an odd number of such extreme $i$-vertex-predecessors, we can conclude that there is a facet going orthogonal through the $i$-hyperplane and $x$ lies on that a facet. Furthermore, since the first $i$-extreme-vertex-predecessor of $x$ lies on facets in all other dimensions $j \neq i$, it holds also for $x$, and thus $x$ is a vertex of the $i$-section.

**Lemma 8.4 (Extreme vertices of a section).** *Let $P$ be an orthogonal polyhedron and let $P' = \mathcal{J}_{i,z}(P)$. A point $x$ is an extreme vertex of $P'$ iff it has an odd number of extreme $i$-vertex-predecessors.*

### 8.3.4 Computing the Intersection

We assume two polyhedra $P_1$ and $P_2$ with $n_1$ and $n_2$ vertices, respectively. The intersection of $P_1$ and $P_2$ may have vertices of $P_1$ and $P_2$ as well as some new vertices, as illustrated in Figure 8.27.

**Lemma 8.5.** *Let $x \in \mathbf{G}$ be a vertex of $P_1 \cap P_2$. Then for every dimension $i$, $x$ is on an $i$-facet of $P_1$ or on an $i$-facet of $P_2$.*

That means, each "new" vertex of the intersection, not being a vertex of $P_1$ or $P_2$, must lie in each dimension $i$ on an $i$-facet of one of the intersecting polyhedra. Since in all three representations we have information about the vertices but not explicitely about the facets, the computation of the intersection of facets must be based on the available information about the vertices.

**Lemma 8.6.** *Let $x$ be a vertex of $P_1 \cap P_2$ which is not an original vertex.*
*Then there exists a vertex $y_1$ of $P_1$ and a vertex $y_2$ of $P_2$ such that $x = \max(y_1, y_2)$, where* max *is applied componentwise.*

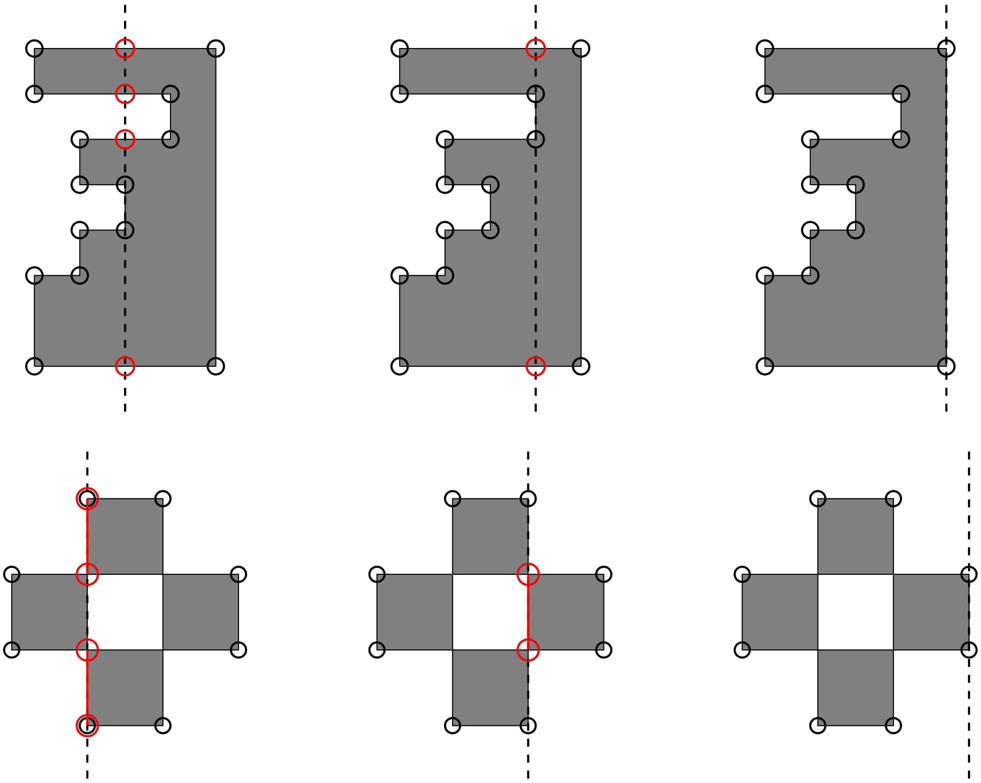


Figure 8.26: Extreme vertices of sections

Note that due to symmetry, we could also use the minimum instead of the maximum in the above lemma.

Thus the candidates for being vertices of $P_1 \cap P_2$ are restricted to members from the set

$$V(P_1) \cup V(P_2) \cup \{x | \exists y_1 \in V(P_1).\ \exists y_2 \in V(P_2).\ x = \max(y_1, y_2)\},$$

whose number is not greater then $n_1 + n_2 + n_1 n_2$.

Figure 8.28 shows those candidates for the example intersection of Figure 8.27.

The algorithm for computating the intersection of two polyhedra $P_1$ and $P_2$ works as follows:

- Initialize $V(P_1) \cup V(P_2)$ as the set of potential vertices of the intersection.
- For every pair of vertices calculate their max and add it to the potential vertex set.
- For each point in the potential vertex set:
  - Compute the color of its neighborhood in both $P_1$ and $P_2$.
  - Calculate the intersection of the neighborhood coloring pointwise.
  - Use the vertex rules to determine, whether the point is a vertex of the intersection.

Remember the vertex rule: A point $x$ is a vertex iff

$$\forall i \in \{1, \ldots, d\}.\ \exists x' \in \mathcal{N}^i(x).\ c(x'^{i-}) \neq c(x').$$

Figure 8.29 shows the computation for our example. The first picture shows the candidates. The next 4 rows illustrate the check of 4 of the candidates. The last row shows the result.

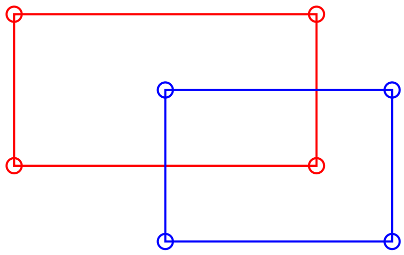


Figure 8.27: The intersection of two orthogonal polyhedra

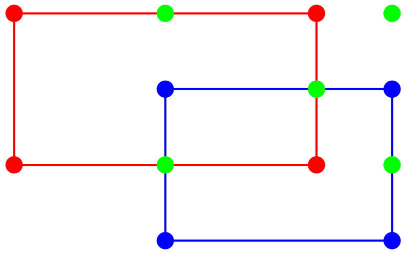


Figure 8.28: Vertex candidates of an intersection of two orthogonal polyhedra

The algorithm for the intersection computation works similarly for all 3 representations. The only difference occurs in the computation of the colors for the neighborhood of the candidates and in the storage of the result.

## Exercises

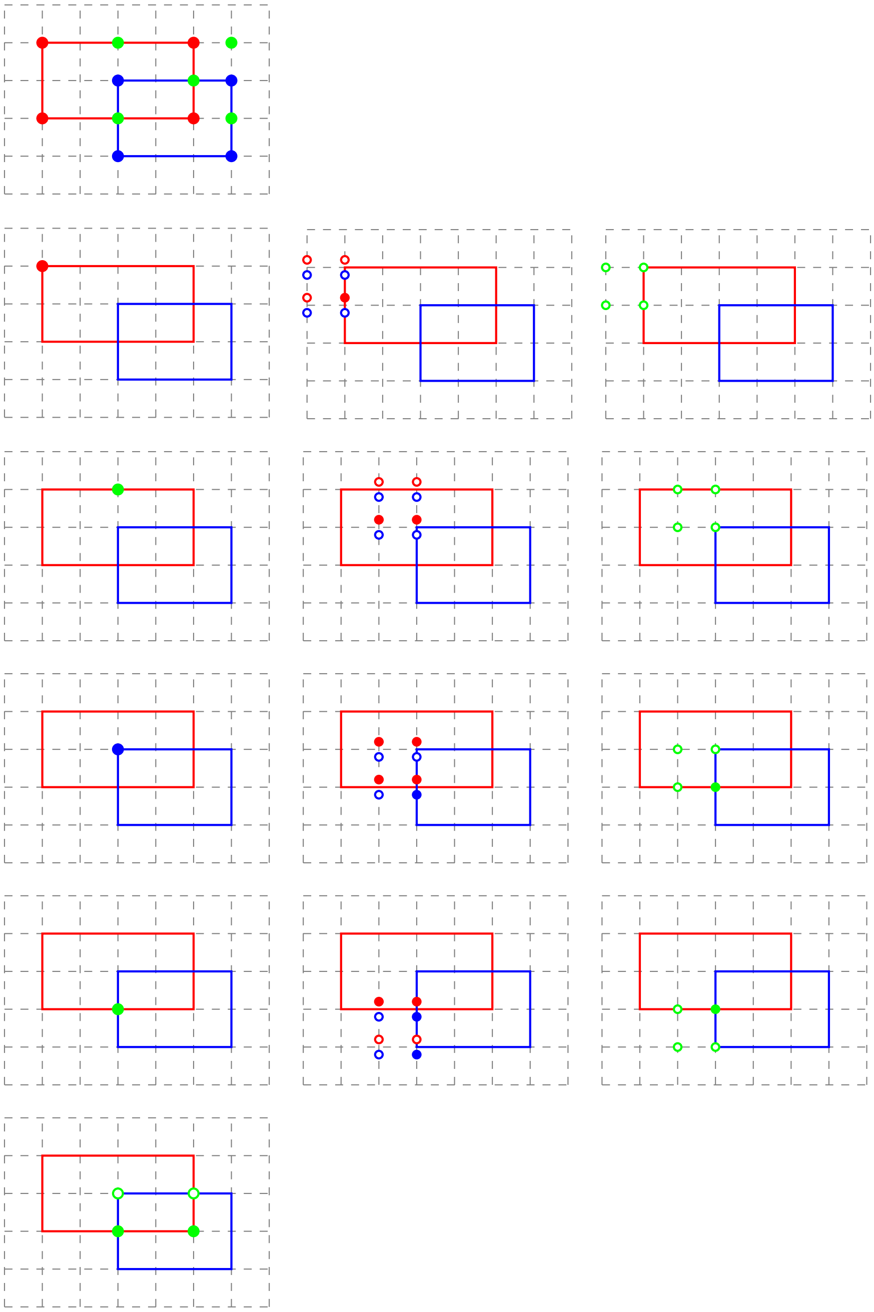


Figure 8.29: Example intersection computation

## 8.4 Convex Polyhedra

After orthogonal polyhedra next we discuss state set representation by *convex polyhedra*. Some polyhedra are depicted in Figure 8.30

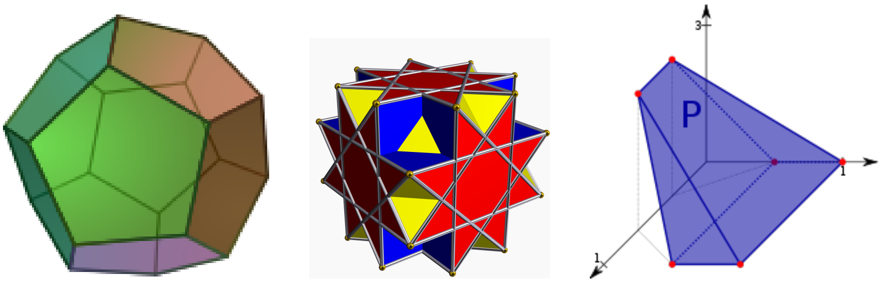


Figure 8.30: Polyhedra

**Definition 8.17.** *A* (convex) polyhedron *in* $\mathbb{R}^d$ *is the solution set to a finite number of inequalities with real coefficients in d real variables. A bounded polyhedron is called* polytope*.*

In the following we restrict ourselves to convex polytopes. An extension to convex polyhedra is possible, but a bit more involved.

We introduce two representation forms for convex polytopes. Defining a polytope by its facets yields an $\mathcal{H}$-representation, whereas the $\mathcal{V}$-representation stores the vertices[1].

**Definition 8.18 (Closed halfspace).** *A d-dimensional* closed halfspace *is a set* $\mathcal{H} = \{x \in \mathbb{R}^d \mid c \cdot x \leq z\}$ *for some* $c \in \mathbb{R}^d$*, called the* normal *of the halfspace, and a* $z \in \mathbb{R}$*.*

**Definition 8.19 ($\mathcal{H}$-polyhedron, $\mathcal{H}$-polytope).** *A d-dimensional* $\mathcal{H}$-polyhedron $P = \bigcap_{i=1}^{n} \mathcal{H}_i$ *is the intersection of finitely many closed halfspaces. A bounded $\mathcal{H}$-polyhedron is called an* $\mathcal{H}$-polytope*.*

The facets of a $d$-dimensional $\mathcal{H}$-polytope are $d-1$-dimensional $\mathcal{H}$-polytopes.

An $\mathcal{H}$-polytope

$$P = \bigcap_{i=1}^{n} \mathcal{H}_i = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^d \mid c_i \cdot x \leq z_i\}$$

can also be written in the form

$$P = \{x \in \mathbb{R}^d \mid Cx \leq z\}.$$

We call $(C, z)$ the *$\mathcal{H}$-representation* of the polytope. Each row $c_i$ of $C$ is the normal vector to the $i$th facet of the polytope. Note that each $\mathcal{H}$-polytope $P$ has a finite number of vertices which we denote by $V(P)$.

**Definition 8.20.** *A set S is called* convex*, if*

$$\forall x, y \in S.\ \forall \lambda \in [0,1] \subseteq \mathbb{R}.\ \lambda x + (1-\lambda) y \in S.$$

$\mathcal{H}$-polyhedra are convex sets.

[1] $\mathcal{H}$ stays for halfspace and $\mathcal{V}$ for vertex.

$\mathit{CH}(V)$ **Definition 8.21 (Convex hull).** *Given a set* $V \subseteq \mathbb{R}^d$*, the* convex hull $\mathit{CH}(V)$ *of* $V$ *is the smallest convex set that contains* $V$*.*

For a finite set $V = \{v_1, \ldots, v_n\}$ the convex hull can be computed by

$$\mathit{CH}(V) = \{x \in \mathbb{R}^d \mid \exists \lambda_1, \ldots, \lambda_n \in [0,1] \subseteq \mathbb{R}^d.\ \sum_{i=1}^{n} \lambda_i = 1 \wedge \sum_{i=1}^{n} \lambda_i v_i = x\}.$$

**Definition 8.22 ($\mathcal{V}$-polytope).** *A* $\mathcal{V}$-polytope $P = \mathit{CH}(V)$ *is the convex hull of a finite set* $V \subset \mathbb{R}^d$*. We call* $V$ *the* $\mathcal{V}$-representation *of the polytope.*

Note that all $\mathcal{V}$-polytopes are bounded. Note furthermore that both representations are in general not canonical as they may be non-redundant: The $\mathcal{H}$-representation may contain redundant subsumed inequations, and the $\mathcal{V}$-representation may contain redundant inner points that are not vertices. This implies that there may be different representations of a single polyhedron. Such superfluous data do not pose theoretical problems, but of course increase the effort of computations. Redundant information can be removed by solving (a set of) linear programms.

For each $\mathcal{H}$-polytope, the convex hull of its vertices defines the same set in the form of a $\mathcal{V}$-polytope, and vice versa, each set defined as a $\mathcal{V}$-polytope can be also given as an $\mathcal{H}$-polytope by computing the halfspaces defined by its facets. This is stated by Motzkin's theorem. However, the translations between the $\mathcal{H}$- and the $\mathcal{V}$-representations of polytopes can be exponential in the state space dimension $d$.

Given a convex polytope, the sizes of the $\mathcal{H}$- and $\mathcal{V}$-representations can strongly differ. For example, on the one hand the $d$-dimensional cube

$$\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid \forall 1 \leq i \leq d.\ -1 \leq x_i \leq 1\}$$

has $2d$ facets and $2^d$ vertices. On the other hand, the $d$-dimensional crosspolytope

$$\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid \sum_{i=1}^{d} |x_i| \leq 1\}$$

has $2d$ vertices and $2^d$ facets.

If we represent reachable sets of hybrid automata by polytopes, we again need certain *operations* on convex polytopes. In the following we discuss

- the *membership* problem,
- the *intersection*, and the
- the *union* of two polytopes.

As we will see, the computations have different complexities in the different representations. Many operations are easily solvable in one of the representation and hard in the other one and vice versa. One could think of converting polytopes for each needed operation into the representation for which the operation is cheap (indeed this is sometimes done). However, note that the conversion can have exponential costs.

- The *membership* problem can be solved in linear time in $d$ in the $\mathcal{H}$-representation. Given an $\mathcal{H}$-polytope defined by $Cx \leq z$ and a point $p \in \mathbb{R}^d$, to check if $p$ is contained in the polytope just substitute $p$ for $x$ in $Cx \leq z$ to check if the inequation holds.

For the $\mathcal{V}$-representation we have to solve a linear programming problem. Given a $\mathcal{V}$-polytope defined by the vertex set $V$, we have to check satisfiability of

$$\exists \lambda_1, \ldots, \lambda_n \in [0,1] \subseteq \mathbb{R}^d . \sum_{i=1}^{n} \lambda_i = 1 \wedge \sum_{i=1}^{n} \lambda_i v_i = x \ .$$

Alternatively we can also convert the $\mathcal{V}$-polytope into an $\mathcal{H}$-polytope by computing its facets and check membership for the $\mathcal{H}$-representation.

- The *intersection* for two polytopes $P_1$ and $P_2$ in the $\mathcal{H}$-representation is again cheap: Given an $\mathcal{H}$-polytope defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$, their intersection is represented by the $\mathcal{H}$-polytope with $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} x \leq \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Note that the resulting representation is in general not minimal.
  Again, the intersection computation for the $\mathcal{V}$-representation is more complex (NP-hard); we do not discuss it here. Assume two $\mathcal{V}$-polytopes $P_1$ and $P_2$ having the vertex sets $V_1$ respectively $V_2$. We can convert the polytopes to $\mathcal{H}$-polytopes, compute their intersection, and convert the result back to a $\mathcal{V}$-polytope.
- For the union, note that the union of two convex polytopes is in general not a convex polytope. The standard way to make the union computation closed under convex polytopes is to take the convex hull of the union.
  This time the computation for the $\mathcal{V}$-representation is more efficient. Assume two $\mathcal{V}$-polytopes defined by the vertex sets $V_1$ and $V_2$. The $\mathcal{V}$-representation of their union is given by $V_1 \cup V_2$. Note again that the representation is not redundant (however, it can be made minimal with additional effort).
  To compute the union of two $\mathcal{H}$-polytopes defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$ is more complex (NP-hard), and we do not handle it here. Alternatively we can convert the polytopes to $\mathcal{V}$-polytopes, compute the union, and compute back the result.

# Exercises

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