Modeling and Analysis of Hybrid Systems

Approximation of reachable state sets

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Informatik 2 - Theory of Hybrid Systems
RWTH Aachen University

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Literatur

Alongkrit Chutinan and Bruce H. Krogh:
Computing Polyhedral Approximations to Flow Pipes for Dynamic Systems
In Proceedings of the 37rd IEEE Conference on Decision and Control, 1998

Olaf Stursberg and Bruce H. Krogh:
Efficient Representation and Computation of Reachable Sets for Hybrid Systems
Hybrid Systems: Computation and Control, LNCS 2623, pp. 482-497, 2003

We had a look at state set approximations by

- convex polyhedra,

and at the basic operations

- testing for membership,
- intersection, and
- union

on these.

Thus we can

- approximate state sets and
- compute with them.

How is all this used in the reachability analysis procedure?

General reachability procedure

Input: Set Init of initial states.

Algorithm:

$$
\begin{array}{l}
R^{\text{new}} := \text{Init}; \\
R := \emptyset; \\
\text{while } (R^{\text{new}} \neq \emptyset)\{ \\
\qquad R \quad := R \cup R^{\text{new}}; \\
\qquad R^{\text{new}} := \boxed{\text{Reach}}(R^{\text{new}}) \backslash R; \\
\}
\end{array}
$$

Output: Set R of reachable states.

What is "Reach"?

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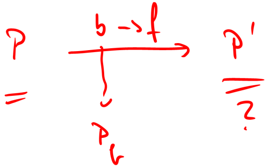
For hybrid systems, independently of the exact definition of "Reach", it will involve the following computations:

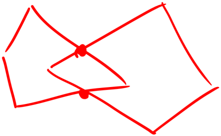
Given a state set R, compute

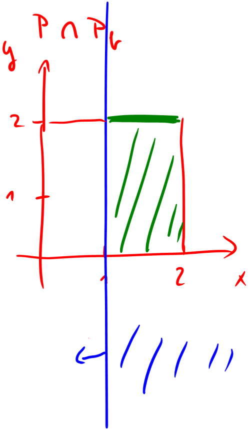
- the set of states reachable from R by a flow (i.e., time transisiton), and
- the set of states reachable from R by a jump (i.e., discrete transition).

Computing the jump successors, i.e., the flow pipe, of a set can be done with the operations we already introduced.

The harder part is computing the flow successors. So let's have a look at that...







$b: \; x \geq 1$

$f: \; y := 2$

$y := x + 2$

Approximating a flow pipe

Consider a dynamical system with state equation

$$\dot{x} = f(x(t)).$$

$\dot{x} = 1$

$\dot{x} \in [a, b]$ $\quad a, b \in \mathbb{R}\ (\mathbb{Q})$

$\dot{x} = Ax + Bu$

$\dot{x} = f(x(t))$

x^2

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Lipschitz continuity implies the existence and uniqueness of the solution to an initial value problem, i.e., for every initial state x_0 there is a unique solution $x(t, x_0)$ to the state equation.

Approximating a flow pipe

The set of reachable states at time t from a set of initial states X_0 is defined as

$$\mathcal{R}_t(X_0) = \{x_f \mid \exists x_0 \in X_0.\ x_f = x(t, x_0)\}.$$

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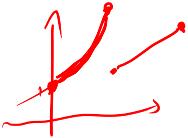
We describe a solution which approximates the flow pipe by a sequence of convex polytopes.



$\dot{x} = Ax\ (+Bu)$

$\dot{x} = x$

Definition (Convex polytope)

Let $POLY(C, d)$ denote the convex polytope defined by the pair $(C, d) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m$ according to

$$POLY(C, d) = \{x \mid Cx \leq d\}.$$

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- Given a finite set of points Γ, the convex hull $CH(\Gamma)$ of Γ is the smallest convex set that contains Γ.

Problem statement for polyhedral approximation of flow pipes

Given

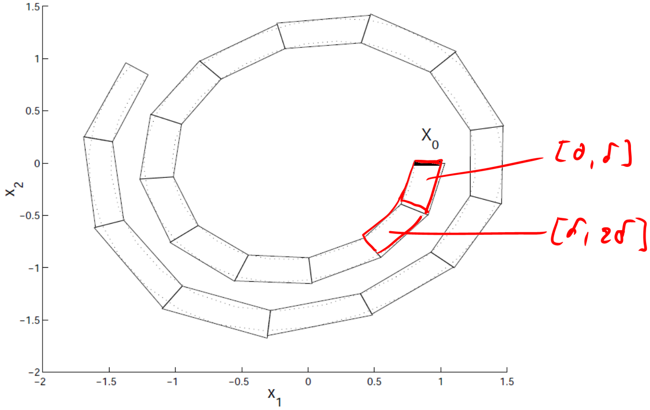
- a set X_0 of initial states which is a polytope, and
- a final time t_f,

compute a polyhedral approximation $\hat{\mathcal{R}}_{[0,t_f]}(X_0)$ to the flow pipe $\mathcal{R}_{[0,t_f]}(X_0)$ such that

$$\mathcal{R}_{[0,t_f]}(X_0) \subseteq \hat{\mathcal{R}}_{[0,t_f]}(X_0).$$

Flow pipe segmentation

Since a single convex polyhedra would strongly overapproximate the flow pipe, we compute a sequence of convex polyhedra, each approximating a flow pipe segment.



Segmented flow pipe approximation

Let the time interval $[0, t_f]$ be divided into $0 < N \in \mathbb{N}$ time segments

$$[0, t_1],\ [t_1, t_2],\ \ldots,\ [t_{N-1}, t_f]$$

with $t_i = i \cdot \frac{t_f}{N}$.

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We generate an approximation $\hat{\mathcal{R}}_{[t_1,t_2]}(X_0)$ for each flow pipe segment:

$$\mathcal{R}_{[t_1,t_2]}(X_0) \subseteq \hat{\mathcal{R}}_{[t_1,t_2]}(X_0).$$

$R_{[0,t_f]}$

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$$\mathcal{R}_{[t_1,t_2]}(X_0) \subseteq \hat{\mathcal{R}}_{[t_1,t_2]}(X_0).$$

The complete flow pipe approximation is the union of the approximation of all N pipe segments:

$$\mathcal{R}_{[0,t_f]}(X_0) \subseteq \hat{\mathcal{R}}_{[0,t_f]}(X_0) = \bigcup_{k=1,\ldots,N} \hat{\mathcal{R}}_{[t_{k-1},t_k]}(X_0)$$

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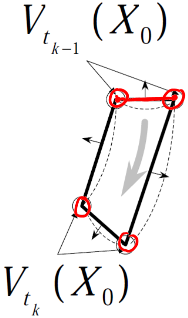

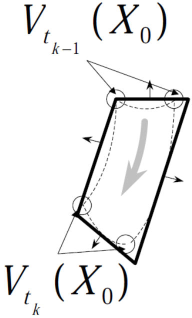

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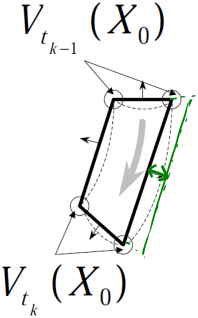

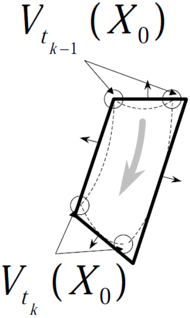

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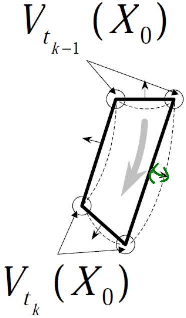
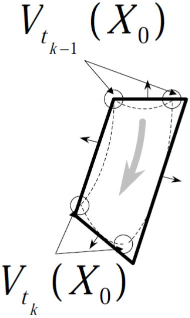
- Evolve vertices: Compute the set of points reachable from the vertices of X_0 in time t_{i-1} and in time t_i.
- Determine hull: Compute the convex hull of those points.
- Bloat hull: Enlarge the hull until it contains all points of the flow pipe segment.



$c_i x \le d_i$



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In particular, we compute the sets $V_{t_{k-1}}(X_0)$ and $V_{t_k}(X_0)$ where

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Each point in the above sets can be obtained

- by analytic solution of the state equation and computing the value, or
- by simulation.

2. Determine hull

We use the evolved vertices in $V_{t_{k-1}}(X_0)$ and $V_{t_k}(X_0)$ to form a convex hull which serves as an initial approximation to the flow pipe segment $\mathcal{R}_{[t_{k-1},t_k]}(X_0)$, denoted by

$$\Phi_{[t_{k-1},t_k]}(X_0) = CH(V_{t_{k-1}}(X_0) \cup V_{t_k}(X_0)).$$

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Let (C_Φ, d_Φ) be the matrix-vector pair defining the convex hull, i.e.,

$$\Phi_{[t_{k-1},t_k]}(X_0) = POLY(C_\Phi, d_\Phi).$$

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- Given: $POLY(C_\Phi, d_\Phi)$.
- We want: $\mathcal{R}_{[t_{k-1},t_k]}(X_0) \subseteq POLY(C_\Phi, \boxed{\text{d}})$.

3. Bloat hull

- We compute d as the solution to the following optimization problem:

$$\min_{d} \quad volume[POLY(C_\Phi, d)] \tag{1}$$
$$s.t. \quad \mathcal{R}_{[t_{k-1}, t_k]}(X_0) \subseteq POLY(C_\Phi, d).$$

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- The ith component d_i^* of the optimum d^* can be found by solving

$$\max_{x} \; c_i^T x \qquad s.t. \; x \in \mathcal{R}_{[t_{k-1},t_k]}(X_0). \tag{2}$$

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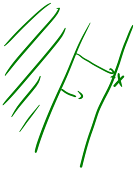
- Solution (x_0^*, t^*) to 3 $\rightarrow$
  Solution $x(t^*, x_0^*)$ to 2 $\rightarrow$
  Solution $d_i^* = c_i^T x(t^*, x_0^*)$ to 1.

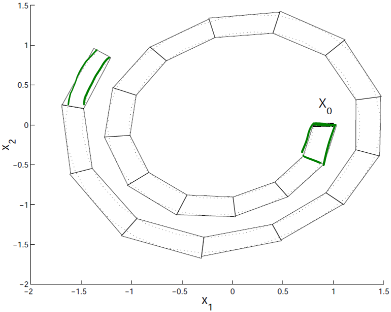


Example

- Van der Pol equation:

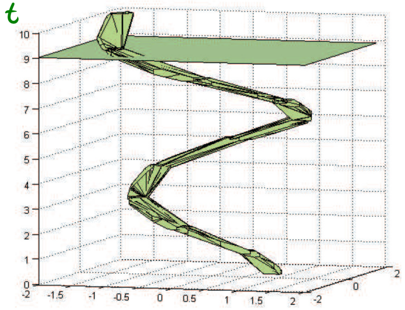
$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.2(x_1^2 - 1)x_2 - x_1.\end{aligned}$$

- Intial set: $X_0 = \{(x_1, x_2) \mid 0.8 \leq x_1 \leq 1 \wedge x_2 = 0\}$.
- Time: $t_f = 10$.
- Segments: 20

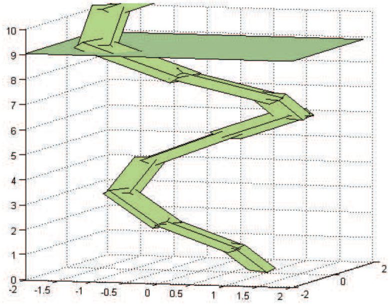


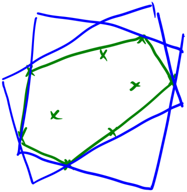
- Van der Pol equation with a third variable being a clock.
- Approximation

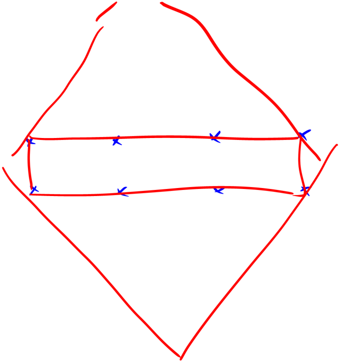
with convex polyhedra and

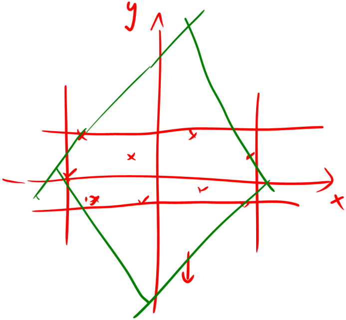


with oriented rectangular hull:








y
x

Partitioning the initial set

Var der Pol system with initial set $X_0 = \{(x_1, x_2) \mid 5 \leq x_1 \leq 45 \wedge x_2 = 0\}$.