Modeling and Analysis of Hybrid Systems

What's decidable about hybrid automata?

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Informatik 2 - Theory of Hybrid Systems
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Literature

Henzinger et al.: What's decidable about hybrid automata?

Journal of Computer and System Sciences, 57:94–124, 1998

Motivation

- The special class of timed automata with TCTL is decidable, thus model checking is possible.
- What about other classes of hybrid systems?

What is decidable about hybrid automata?

Two central problems for the analysis of hybrid automata:

- Safety: The problem to decide if something "bad" can happend during the execution of a system.
- Liveness: The problem to decide if there is always the possibility that something "good" will eventually happen during the execution of a system.

Both problems are decidable in certain special cases, and undecidable in certain general cases.

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A particularly interesting class:

- all conditions, effects, and flows are described by rectagular sets.

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A particularly interesting class:

- all conditions, effects, and flows are described by rectagular sets.

Definition

- A set $\mathcal{R} \subset \mathbb{R}^n$ is rectangular if it is a cartesian product of (possibly unbounded) intervals, all of whose endpoints are rationals.
- The set of rectangular sets in $\mathbb{R}^n$ is denoted $\mathcal{R}^n$.

Rectangular automaton

Definition

A rectangular automaton A is a tuple
$\mathcal{H} = (Loc, Var, Con, Lab, Edge, Act, Inv, Init)$ with

- finite set of locations Loc,
- finite set of real-valued variables $Var = \{x_1, \ldots, x_n\}$,
- a function *Con* : $Loc \to 2^{Var}$ assigning controlled variables to locations,
- finite set of synchronization labels Lab,
- finite set of edges $Edge \subseteq Loc \times Lab \times \mathcal{R}^n \times \mathcal{R}^n \times 2^{\{1,\ldots,n\}} \times Loc$,
- a flow function $Act : Loc \to \mathcal{R}^n$,
- an invariant function $Inv : Loc \to \mathcal{R}^n$,
- initial states $Init : Loc \to \mathcal{R}^n$.

Rectangular automaton with ϵ-moves: Lab contains ϵ (also denoted by τ).

State space

- States: $\sigma = (l, \vec{x}) \in (Loc \times \mathbb{R}^n)$ with $\vec{x} \in Inv(l)$

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- Is the state space rectangular?
- Do the initial states build a rectangular set?

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- State space: $\Sigma \subseteq Loc \times \mathbb{R}^n$ is the set of all states
- Is the state space rectangular?
- Do the initial states build a rectangular set?
- May we use conjunctions to specify the invariants?

Rectangular automaton

- Flows: first time derivatives of the flow trajectories in location $l \in Loc$ are within $Act(l)$
- Jumps: $e = (l, a, pre, post, jump, l') \in Edge$ may move control from location l to location l' starting from a valuation in pre, changing the value of each variable x_i to a nondeterministically chosen value from $post_i$ (the projection of $post$ to the ith dimension), such that the values of the variables $x_i \notin jump$ are unchanged.

Operational semantics

Operational semantics

$$\frac{(l, a, \mathit{pre}, \mathit{post}, \mathit{jump}, l') \in Edge \qquad \vec{x} \in \mathit{pre} \quad \vec{x}' \in \mathit{post} \quad \forall i \notin \mathit{jump}.\ x_i' = x_i \quad \vec{x}' \in Inv(l')}{(l, \vec{x}) \xrightarrow{a} (l', \vec{x}')} \quad \texttt{Rule}_{\texttt{Discrete}}$$

Operational semantics

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$$\frac{(t = 0 \wedge \vec{x} = \vec{x}') \vee (t > 0 \wedge (\vec{x}' - \vec{x})/t \in \mathit{Act}(l)) \quad \vec{x}' \in \mathit{Inv}(l)}{(l, \vec{x}) \xrightarrow{t} (l, \vec{x}')} \ \texttt{Rule}_{\texttt{Time}}$$

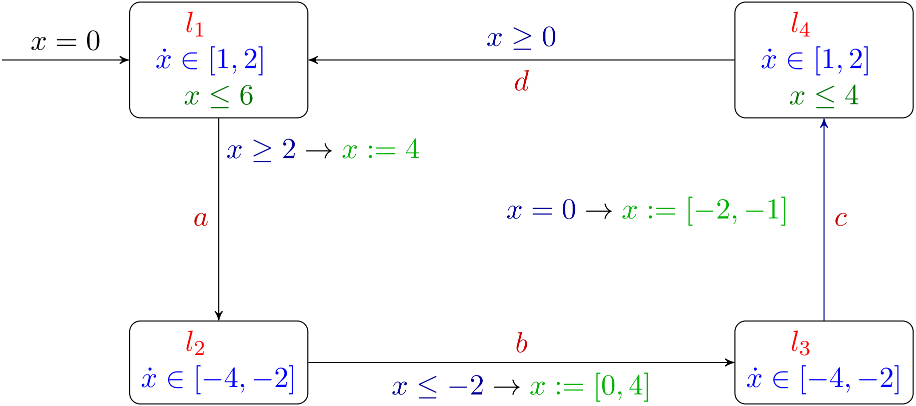
Operational semantics

$$\frac{\begin{array}{c}(l, a, \mathit{pre}, \mathit{post}, \mathit{jump}, l') \in Edge \\ \vec{x} \in \mathit{pre} \quad \vec{x}' \in \mathit{post} \quad \forall i \notin \mathit{jump}.\ x'_i = x_i \quad \vec{x}' \in Inv(l')\end{array}}{(l, \vec{x}) \xrightarrow{a} (l', \vec{x}')} \quad \texttt{Rule}_{\texttt{Discrete}}$$

$$\frac{(t = 0 \wedge \vec{x} = \vec{x}') \vee (t > 0 \wedge (\vec{x}' - \vec{x})/t \in Act(l)) \quad \vec{x}' \in Inv(l)}{(l, \vec{x}) \xrightarrow{t} (l, \vec{x}')} \quad \texttt{Rule}_{\texttt{Time}}$$

- Execution step: $\rightarrow\ = \xrightarrow{a} \cup \xrightarrow{t}$
- Path: $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$
- Initial path: path $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \vec{x}_0)$, $\vec{x}_0 \in Init(l_0) \cap Inv(l_0)$
- Reachability of a state: exists an initial path leading to the state

Initialized rectangular automaton



Remarks

- If we replace rectangular sets with linear sets, we obtain linear hybrid automata, a super-class of rectangular automata.
- A timed automaton is a special rectangular automaton.

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- A timed automaton is a special rectangular automaton.

This class lies at the boundary of decidability.

Decidability

The reachability problem is decidable for initialized rectangular automata:

Decidability

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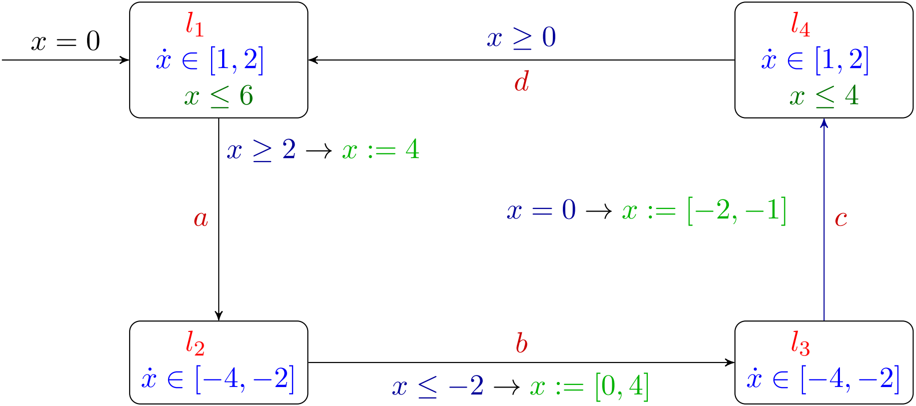
Definition

A rectangular automaton A is initialized, if for every edge $(l, a, pre, post, jump, l')$ of A, and every variable index $i \in \{1, \ldots, n\}$ with $Act(l)_i \neq Act(l')_i$, we have that $i \in jump$.

The reachability problem becomes undecidable if one of the restrictions is relaxed.



This rectangular automaton is initialized.

Decidability results

Lemma

The reachability problem for initialized rectangular automata is complete for PSPACE.

Decidability results

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Timed automaton

$\uparrow$

Initialized stopwatch automaton

$\uparrow$

Initialized singular automaton

$\uparrow$

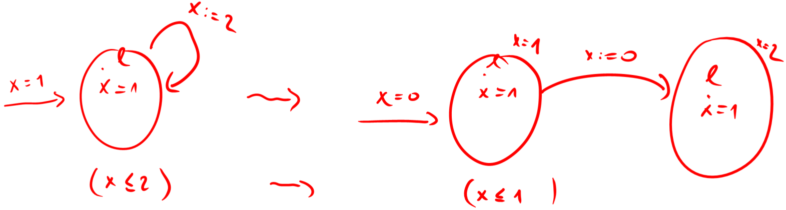
Initialized rectangular automaton

A timed automaton is a rectangular automaton with deterministic jumps, i.e.,

- $Init(l)$ is empty or a singleton for each $l \in Loc$,
- for each edge, *post*$_i$ is a single value for each $i \in$ *jump*,

and every variable is a clock, i.e.,

- $Act(l)(x) = [1, 1]$ for all locations l and variables x.



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Lemma

The reachability problem for timed automata is complete for PSPACE.

Decidability results

Timed automaton

$\uparrow$

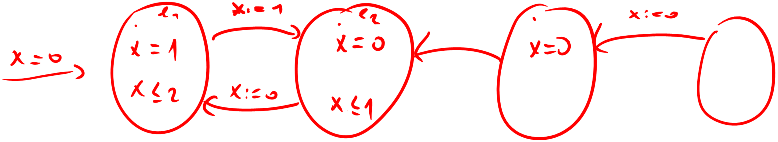
Initialized stopwatch automaton

- A stopwatch is a variable with derivatives 0 or 1 only.
- A stopwatch automaton is a rectangular automaton with deterministic jumps and stopwatch variables only.
- Initialized stopwatch automata can be polynomially encoded by timed automata.

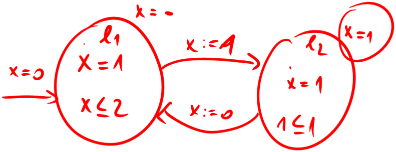
Lemma

The reachability problem for initialized stopwatch automata is complete for PSPACE.

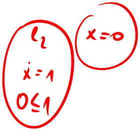
However, the reachability problem for non-initialized stopwatch automata is undecidable.



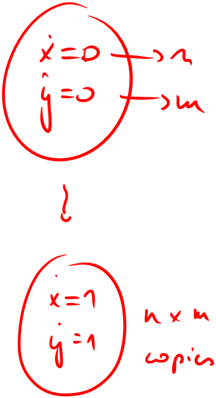
x = 0
l1
ẋ = 1
x ≤ 2
x := 1
x := 0
l2
ẋ = 0
x ≤ 1
ẋ = 0
x := 0



x = 0
x = 0
l1
ẋ = 1
x ≤ 2
x := 1
x := 0
l2
ẋ = 1
1 ≤ 1
x = 1



l2
ẋ = 1
0 ≤ 1
x = 0



ẋ = 0 → n
ẏ = 0 → m
ẋ = 1
ẏ = 1
n × m
copies

Proof idea:
Notice, that a timed automaton is a stopwatch automaton such that every variable is a clock.
Assume that C is an n-dimensional initialized stopwatch automaton. Let κ_C be the set of constants used in the definition of C, and let $\kappa_- = \kappa_C \cup \{-\}$.
We define an n-dimensional timed automaton D_C with locations $Loc_{D_C} = Loc_c \times \kappa_-^{1,\ldots,n}$. Each location (l, f) of D_C consists of a location l of C and a function $f : \{1, \ldots, n\} \to \kappa_-$. Each state $q = ((l, f), \vec{x})$ of D_C represents the state $\alpha(q) = (l, \vec{y})$ of C, where $y_i = x_i$ if $f(i) = -$, and $y_i = f(i)$ if $f(i) \neq -$.
Intuitively, if the ith stopwatch of C is running (slope 1), then its value is tracked by the value of the ith clock of D_C; if the ith stopwatch is halted (slope 0) at value $k \in \kappa_C$, then this value is remembered by the current location of D_C.

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

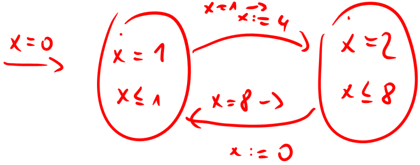
$\uparrow$

Initialized singular automaton

- A variable x_i is a finite-slope variable if $flow(l)_i$ is a singleton in all locations l.
- A singular automaton is a rectangular automaton with deterministic jumps such that every variable of the automaton is a finite-slope variable.
- Initialized singular automata can be rescaled to initialized stopwatch automata.

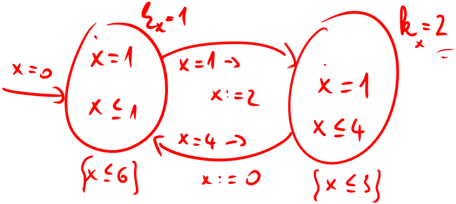
Lemma

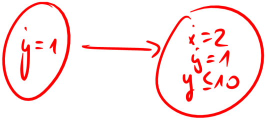
The reachability problem for initialized singular automata is complete for PSPACE.

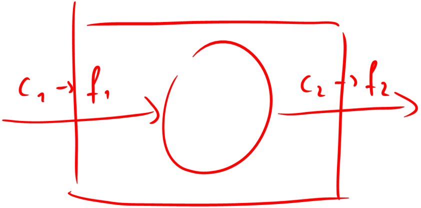

x = 0
x = 1
x ≤ 1
x = 1 → x := 4
x = 2
x ≤ 8
x = 8 →
x := 0

EF x ≤ 6 ~>


n/2
n


k_x = 1
k_x = 2
x = 0
x = 1
x ≤ 1
x = 1 →
x := 2
x = 1
x ≤ 4
x = 4 →
x := 0
{x ≤ 6}
{x ≤ 3}


y = 1
x = 2
y = 1
y ≤ 10


c₁ → f₁
c₂ → f₂

Proof idea: Let B be an n-dimensional initialized singular automaton. We define an n-dimensional initialized stopwatch automaton C_B with the same location set, edge set, and label set as B.
Each state $q = (l, \vec{x})$ of C_B corresponds to the state $\beta(q) = (l, \beta(\vec{x}))$ of B with $\beta : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows:
For each location l of B, if $Act_B(l) = \Pi_{i=1}^{n}[k_i, k_i]$, then $\beta(x_1, \ldots, x_n) = (l_1 \cdot x_1, \ldots, l_n \cdot x_n)$ with $l_i = k_i$ if $k_i \neq 0$, and $l_i = 1$ if $k_i = 0$;
β can be viewed as a rescaling of the state space. All conditions in the automaton B occur accordingly rescaled in C_B.
We have:

- The reachable set of $Reach(B)$ of B is $\beta(Reach(C_B))$.
- $Lang(B) = Lang(C_B)$

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

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Initialized singular automaton

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Initialized rectangular automaton

Lemma

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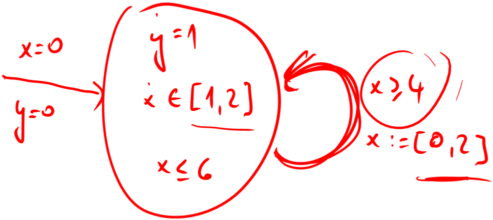
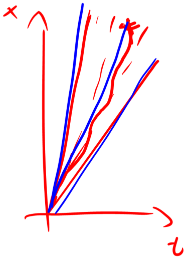
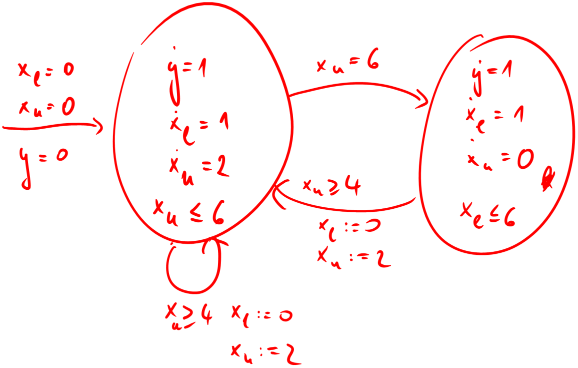


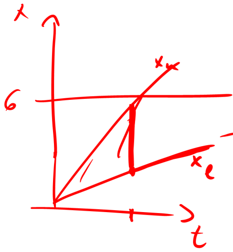
$x = 0$
$y = 0$
$\dot{y} = 1$
$\dot{x} \in [1,2]$
$x \leq 6$
$x \geq 4$
$x := [0,2]$



x
t



$x_l = 0$
$x_u = 0$
$y = 0$
$\dot{y} = 1$
$\dot{x}_l = 1$
$\dot{x}_u = 2$
$x_u \leq 6$
$x_u = 6$
$\dot{y} = 1$
$\dot{x}_l = 1$
$\dot{x}_u = 0$
$x_l \leq 6$
$x_u \geq 4$
$x_l := 0$
$x_u := 2$
$x_u \geq 4$ $x_l := 0$
$x_u := 2$



x
6
x_u
x_l
t

Proof idea: An n-dimensional initialized rectangular automaton A can be translated into a $(2n+1)$-dimensional initialized singular automaton B, such that B contains all reachability information about A.

The translation is similar to the subset construction for determinizing finite automata.

The idea is to replace each variable c of A by two finite-slope variables c_l and c_u: the variable c_l tracks the least possible value of c, and c_u tracks the greatest possible value of c.