Modeling and Analysis of Hybrid Systems

What's decidable about hybrid automata?

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Informatik 2 - Theory of Hybrid Systems
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Literature

Henzinger et al.: What's decidable about hybrid automata?

Journal of Computer and System Sciences, 57:94–124, 1998

Motivation

- The special class of timed automata with TCTL is decidable, thus model checking is possible.
- What about more expressive model classes for hybrid systems?

What is decidable about hybrid automata?

Two central problems for the analysis of hybrid automata:

- Safety: The problem to decide whether something “bad” can happend during the execution of a system.
- Liveness: The problem to decide whether there is always the possibility that something “good” will eventually happen during the execution of a system.

Both problems are decidable in certain special cases, and undecidable in certain general cases.

What is decidable about hybrid automata?

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- all conditions, effects, and flows are described by rectagular sets.

What is decidable about hybrid automata?

A particularly interesting class:

- all conditions, effects, and flows are described by rectagular sets.

Definition

- A set $\mathcal{R} \subset \mathbb{R}^n$ is rectangular if it is a cartesian product of (possibly unbounded) intervals, all of whose endpoints are rationals.
- The set of rectangular sets in $\mathbb{R}^n$ is denoted $\underline{\mathcal{R}^n}$.

$$\Sigma = Loc \times \underline{\underline{\mathbb{R}^n}}$$

Rectangular automaton

Definition

A rectangular automaton A is a tuple
$\mathcal{H} = (Loc, Var, Con, Lab, Edge, Act, Inv, Init)$ with

- finite set of locations Loc,
- finite set of real-valued variables $Var = \{x_1, \ldots, x_n\}$,
- a function **Con** $: Loc \to 2^{Var}$ assigning controlled variables to locations,
- finite set of synchronization labels Lab,
- finite set of edges $Edge \subseteq Loc \times Lab \times \underbrace{\mathcal{R}^n \times \mathcal{R}^n \times 2^{\{1,\ldots,n\}}} \times Loc$,
- a flow function $Act : Loc \to \mathcal{R}^n$,
- an invariant function $Inv : Loc \to \mathcal{R}^n$,
- initial states $Init : Loc \to \mathcal{R}^n$.

$x := x + 1$

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- States: $\sigma = (l, \vec{x}) \in (Loc \times \mathbb{R}^n)$ with $\vec{x} \in Inv(l)$

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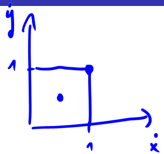
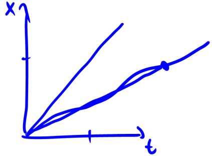
- State space: $\Sigma \subseteq Loc \times \mathbb{R}^n$ is the set of all states
- Is the state space rectangular?

x_i

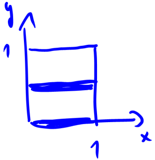
$\vec{x} = (x_1, \ldots, x_n)$

Rectangular automaton





- Flows: first time derivatives of the flow trajectories in location $l \in Loc$ are within $Act(l)$
- Jumps: $e = (l, a, pre, post, jump, l') \in Edge$ may move control from location l to location l' starting from a valuation in pre, changing the value of each variable x_i to a nondeterministically chosen value from $post_i$ (the projection of $post$ to the ith dimension), such that the values of the variables $x_i \notin jump$ are unchanged.



$jump = \{x\}$

Operational semantics

Operational semantics

$$\frac{(l, a, \mathit{pre}, \mathit{post}, \mathit{jump}, l') \in Edge \qquad \vec{x} \in \mathit{pre} \quad \vec{x}' \in \mathit{post} \quad \boxed{\forall i \notin \mathit{jump}.\ x'_i = x_i} \quad \vec{x}' \in Inv(l')}{(l, \underline{\vec{x}}) \xrightarrow{a} (l', \underline{\vec{x}'})} \quad \texttt{Rule}_{\texttt{Discrete}}$$

Operational semantics

$$\frac{\begin{array}{c}(l, a, \mathit{pre}, \mathit{post}, \mathit{jump}, l') \in Edge \\ \vec{x} \in \mathit{pre} \quad \vec{x}' \in \mathit{post} \quad \forall i \notin \mathit{jump}.\ x_i' = x_i \quad \vec{x}' \in Inv(l')\end{array}}{(l, \vec{x}) \xrightarrow{a} (l', \vec{x}')} \quad \texttt{Rule}_{\texttt{Discrete}}$$

$$\frac{(t = 0 \wedge \vec{x} = \vec{x}') \vee (t > 0 \wedge (\vec{x}' - \vec{x})/t \in Act(l)) \quad \vec{x}' \in Inv(l)}{(l, \vec{x}) \xrightarrow{t} (l, \vec{x}')} \quad \texttt{Rule}_{\texttt{Time}}$$

Operational semantics

$$\frac{\begin{array}{c}(l, a, \mathit{pre}, \mathit{post}, \mathit{jump}, l') \in \mathit{Edge} \\ \vec{x} \in \mathit{pre} \quad \vec{x}' \in \mathit{post} \quad \forall i \notin \mathit{jump}.\ x'_i = x_i \quad \vec{x}' \in \mathit{Inv}(l')\end{array}}{(l, \vec{x}) \xrightarrow{a} (l', \vec{x}')} \ \texttt{Rule}_{\texttt{Discrete}}$$

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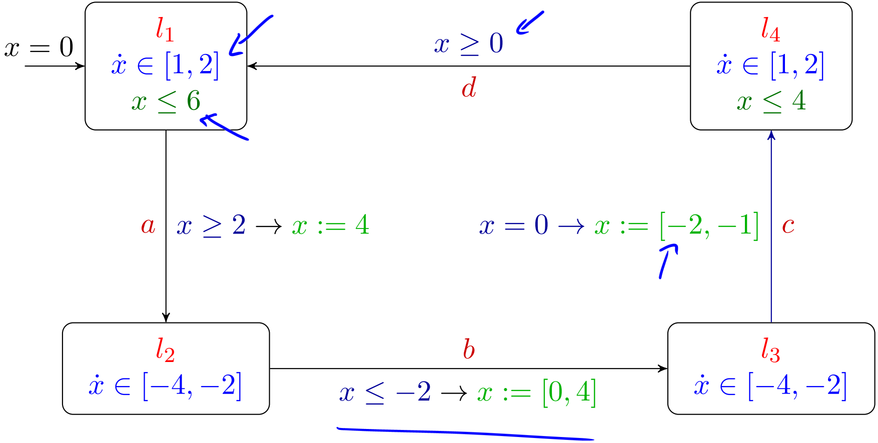
- Execution step: $\rightarrow \; = \; \xrightarrow{a} \cup \xrightarrow{t}$
- Path: $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \vec{x}_0)$, $\vec{x}_0 \in \mathit{Inv}(l_0)$
- Initial path: path $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \vec{x}_0)$, $\vec{x}_0 \in \mathit{Init}(l_0)$
- Reachability of a state: exists an initial path leading to the state


$x = 0$
l_1
$\dot{x} \in [1, 2]$
$x \leq 6$
$x \geq 0$
d
l_4
$\dot{x} \in [1, 2]$
$x \leq 4$
a
$x \geq 2 \rightarrow x := 4$
$x = 0 \rightarrow x := [-2, -1]$
c
l_2
$\dot{x} \in [-4, -2]$
b
$x \leq -2 \rightarrow x := [0, 4]$
l_3
$\dot{x} \in [-4, -2]$

Remarks

- If we replace rectangular sets with linear sets, we obtain linear hybrid automata, a super-class of rectangular automata.
- A timed automaton is a special rectangular automaton.

$3x + y \leq 2$

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- If we replace rectangular sets with linear sets, we obtain linear hybrid automata, a super-class of rectangular automata.
- A timed automaton is a special rectangular automaton.

This class lies at the boundary of decidability.

Decidability

The reachability problem is decidable for initialized rectangular automata:

Decidability

The reachability problem is decidable for initialized rectangular automata:

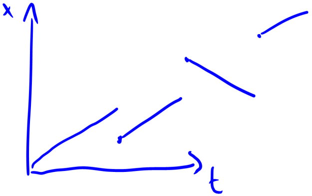
Definition

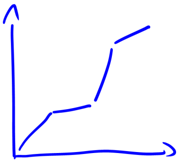
A rectangular automaton A is initialized, if for every edge $(l, a, pre, post, jump, l')$ of A, and every variable index $i \in \{1, \ldots, n\}$ with $Act(l)_i \neq Act(l')_i$, we have that $i \in jump$.

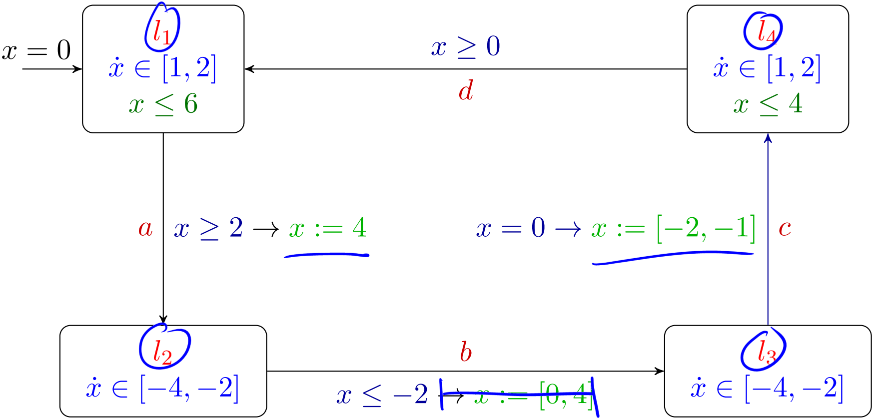
The reachability problem becomes undecidable if one of the restrictions is relaxed.

$x := x$

$x := [a, b]$







This rectangular automaton is initialized.

What we already know

A timed automaton is a special rectangular automaton such that

- for each edge, *post*$_i$ is a single value for each $i \in$ *jump* and
- every variable is a clock, i.e., $Act(l)(x) = [1, 1]$ for all locations l and variables x.

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Lemma

The reachability problem for timed automata is complete for PSPACE.

Decidability results

Lemma

The reachability problem for initialized rectangular automata is complete for PSPACE.

Decidability results

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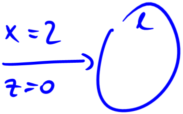
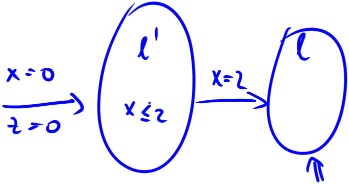
The reachability problem for initialized rectangular automata is complete for PSPACE.

$$\begin{array}{c}\text{Timed automaton}\\ \uparrow \\ \text{Initialized stopwatch automaton}\\ \uparrow \\ \text{Initialized singular automaton}\\ \uparrow \\ \text{Initialized rectangular automaton}\end{array}$$





$$(\ell, v, z) \approx (\ell, v, z+2)$$

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

Initialized stopwatch automata

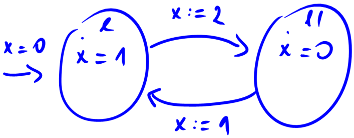
- A stopwatch is a variable with derivatives 0 or 1 only.
- A stopwatch automaton is as a timed automaton but allowing stopwatch variables instead of clocks.
- Initialized stopwatch automata can be polynomially encoded by timed automata.

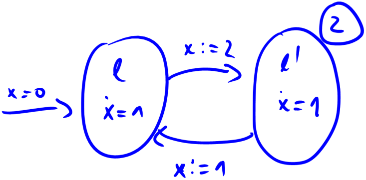
Lemma

The reachability problem for initialized stopwatch automata is complete for PSPACE.

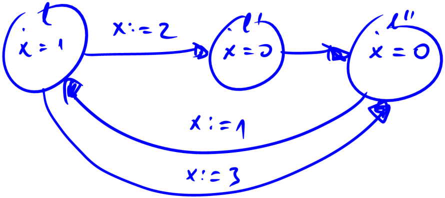
However, the reachability problem for non-initialized stopwatch automata is undecidable.

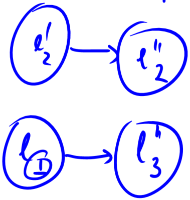
Proof idea:





$$(\ell', x=2) = f((\ell', \nu))$$





Proof idea: Notice, that a timed automaton is a stopwatch automaton such that every variable is a clock.
Assume that C is an n-dimensional initialized stopwatch automaton. Let κ_C be the set of constants used in the definition of C, and let $\kappa_- = \kappa_C \cup \{-\}$.
We define an n-dimensional timed automaton D_C with locations $Loc_{D_C} = Loc_c \times \kappa_-^{1,\dots,n}$. Each location (l, f) of D_C consists of a location l of C and a function $f : \{1, \dots, n\} \to \kappa_-$. Each state $q = ((l, f), \vec{x})$ of D_C represents the state $\alpha(q) = (l, \vec{y})$ of C, where $y_i = x_i$ if $f(i) = -$, and $y_i = f(i)$ if $f(i) \neq -$.
Intuitively, if the ith stopwatch of C is running (slope 1), then its value is tracked by the value of the ith clock of D_C; if the ith stopwatch is halted (slope 0) at value $k \in \kappa_C$, then this value is remembered by the current location of D_C.

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

$\uparrow$

Initialized singular automaton

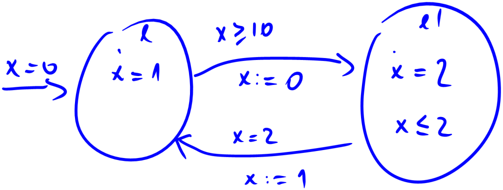
Initialized singular automata

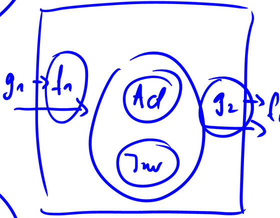
- A variable x_i is a finite-slope variable if $\mathit{flow}(l)_i$ is a singleton in all locations l.
- A singular automaton is as a stopwatch automaton but allowing finite-slope variables instead of stopwatches.
- Initialized singular automata can be polynomially encoded by initialized stopwatch automata.

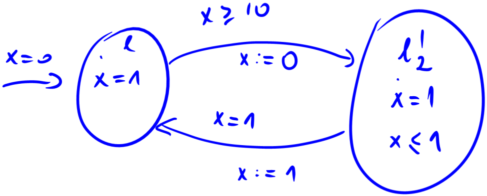
Lemma

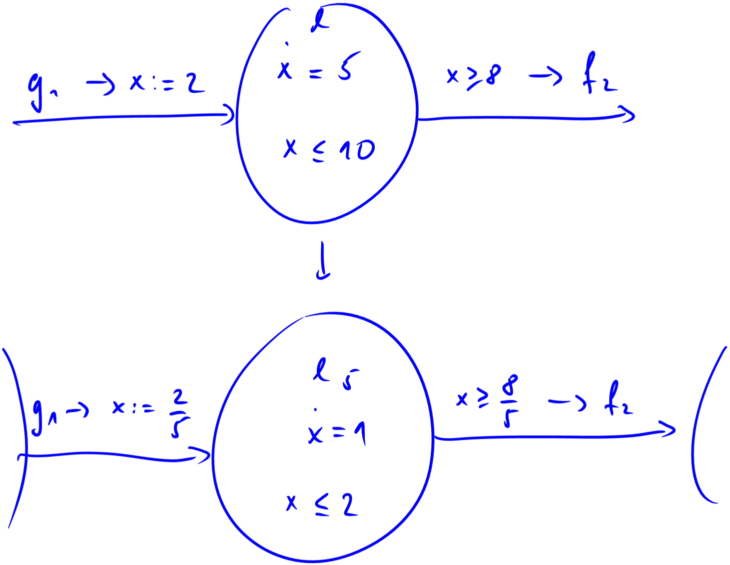
The reachability problem for initialized singular automata is complete for PSPACE.

Proof idea:








$g_1 \to x := 2$
ℓ
$\dot{x} = 5$
$x \leq 10$
$x \geq 8 \to f_2$
$g_1 \to x := \frac{2}{5}$
ℓ_5
$\dot{x} = 1$
$x \leq 2$
$x \geq \frac{8}{5} \to f_2$

Proof idea: Let B be an n-dimensional initialized singular automaton. We define an n-dimensional initialized stopwatch automaton C_B with the same location set, edge set, and label set as B.

Each state $q = (l, \vec{x})$ of C_B corresponds to the state $\beta(q) = (l, \beta(\vec{x}))$ of B with $\beta : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows:

For each location l of B, if $Act_B(l) = \Pi_{i=1}^{n}[k_i, k_i]$, then $\beta(x_1, \ldots, x_n) = (l_1 \cdot x_1, \ldots, l_n \cdot x_n)$ with $l_i = k_i$ if $k_i \neq 0$, and $l_i = 1$ if $k_i = 0$;

β can be viewed as a rescaling of the state space. All conditions in the automaton B occur accordingly rescaled in C_B.

We have:

- The reachable set of $Reach(B)$ of B is $\beta(Reach(C_B))$.

Decidability results

Timed automaton

$\uparrow$

Initialized stopwatch automaton

$\uparrow$

Initialized singular automaton

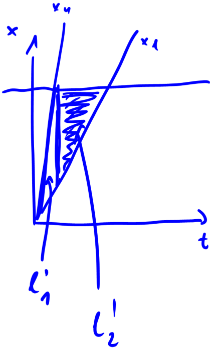
$\uparrow$

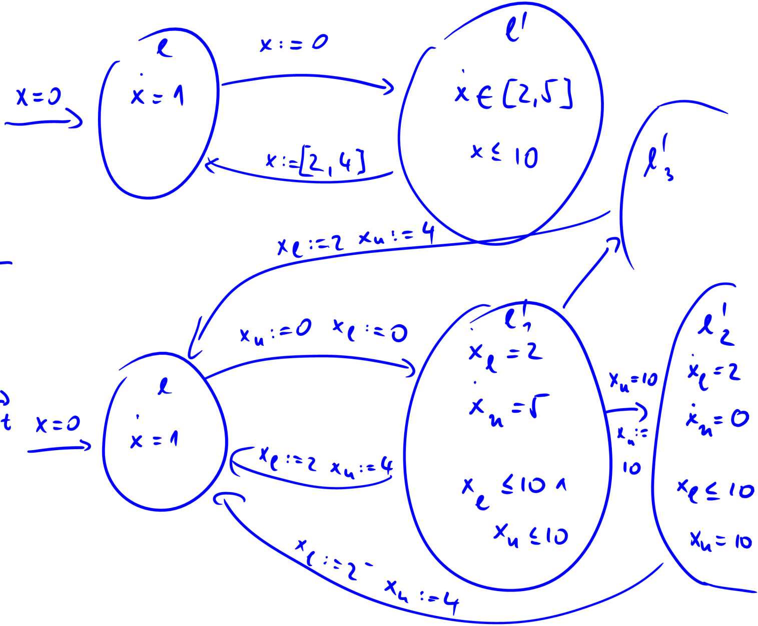
Initialized rectangular automaton

Lemma

The reachability problem for initialized rectangular automata is complete for PSPACE.

Proof idea:


x
x_u
x_l
t
l'_1
l'_2


x=0
l
$\dot{x}=1$
x:=0
l'
$\dot{x} \in [2,5]$
$x \leq 10$
x:=[2,4]
l'_3
$x_l := 2$ $x_u := 4$
$x_u := 0$ $x_l := 0$
x=0
l
$\dot{x}=1$
l'_1
$\dot{x}_l = 2$
$\dot{x}_u = 5$
$x_l \leq 10 \wedge$
$x_u \leq 10$
$x_l := 2$ $x_u := 4$
$x_u = 10$
$x_u := 10$
l'_2
$\dot{x}_l = 2$
$\dot{x}_u = 0$
$x_l \leq 10$
$x_u = 10$
$x_l := 2$ $x_u := 4$

Proof idea: An n-dimensional initialized rectangular automaton A can be translated into a $2n$-dimensional initialized singular automaton B, such that B contains all reachability information about A.
The translation is similar to the subset construction for determinizing finite automata.
The idea is to replace each variable c of A by two finite-slope variables c_l and c_u: the variable c_l tracks the least possible value of c, and c_u tracks the greatest possible value of c.