Modeling and Analysis of Hybrid Systems

Timed automata

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Informatik 2 - Theory of Hybrid Systems
RWTH Aachen University

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Literature

Christel Baier and Joost-Pieter Katoen:
Principles of Model Checking

Contents

Motivation

Correctness in time-critical systems not only depends on the logical result of the computation but also on the time at which the results are produced.

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Thus if we model such systems, we also need to model the time.
The first choice in modeling: discrete or continuous time?

Discrete-time systems

Discrete-time systems

Advantages:

- conceptually simple
- each action lasts for a single time unit (tick)
- action α lasts $k > 0$ time units $\rightsquigarrow$ $k - 1$ ticks followed by α

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Logic: CTL or LTL extended with syntactic sugar

$\mathcal{X}\varphi$: φ holds after one tick

$\mathcal{X}^k\varphi$: φ holds after k ticks

$\mathcal{F}^{\leq k}\varphi$: φ occurs within k ticks

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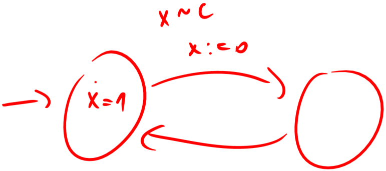
We deal in this lecture with continuous-time models.

Contents

Timed automata

- Measure time: finite set $\mathcal{C}$ of clocks $x, y, z, \ldots$
- Clocks increase their value implicitely as time progresses
- All clocks proceed at rate 1

$\sim \in \{\leq, <, =, >, \geq\}$



LTS

C

Timed automata

- Measure time: finite set $\mathcal{C}$ of clocks $x, y, z, \ldots$
- Clocks increase their value implicitely as time progresses
- All clocks proceed at rate 1
- Limited clock access:

Reading: Clock constraints

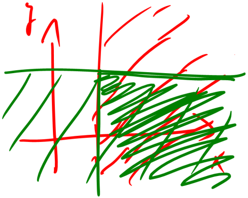
$$g \quad ::= \quad x < c \quad | \quad x \leq c \quad | \quad x > c \quad | \quad x \geq c \quad | \quad g \wedge g$$

with $c \in \mathbb{N}$ ($c \in \mathbb{Q}$) and $x \in \mathcal{C}$.

Syntactic sugar: *true*, $\quad x \in [c_1, c_2), \quad c_1 \leq x < c_2, \quad x = c, \ldots$

$ACC(\mathcal{C})$: set of atomic clock constraints over $\mathcal{C}$
$CC(\mathcal{C})$: set of clock constraints over $\mathcal{C}$

Writing: Clock reset sets value to 0



Semantics of clock constraints

Given a set $\mathcal{C}$ of clocks, a clock valuation $\nu : \mathcal{C} \to \mathbb{R}_{\geq 0}$ assigns a non-negative value to each clock. We use $V_{\mathcal{C}}$ to denote the set of clock valuations for the clock set $\mathcal{C}$.

Definition

For a set $\mathcal{C}$ of clocks, $x \in \mathcal{C}$, $\nu \in V_{\mathcal{C}}$, $c \in \mathbb{N}$, and $g, g' \in CC(\mathcal{C})$, let $\models \; \subseteq \; V_{\mathcal{C}} \times CC(\mathcal{C})$ be defined by

$$
\begin{array}{lll}
\nu \models x < c & \text{iff} & \nu(x) < c \\
\nu \models x \leq c & \text{iff} & \nu(x) \leq c \\
\nu \models x > c & \text{iff} & \nu(x) > c \\
\nu \models x \geq c & \text{iff} & \nu(x) \geq c \\
\nu \models g \wedge g' & \text{iff} & \nu \models g \text{ and } \nu \models g'
\end{array}
$$

Semantics of clock access

Definition

- For a set $\mathcal{C}$ of clocks, $\nu \in V_{\mathcal{C}}$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in \mathcal{C}$.
- For a valuation $\nu \in V_{\mathcal{C}}$ and a clock $x \in \mathcal{C}$ we define *reset x in ν* to be the valuation which equals ν except on x whose value is 0:

$$(\textit{reset } x \textit{ in } \nu)(y) = \begin{cases} \nu(y) & \textit{if } y \neq x \\ 0 & \textit{else} \end{cases}$$

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What does it mean?

- $(\nu + 9)(x) = \nu(x) + 9$

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What does it mean?

- $\nu + 9$
- $\textit{reset } x \textit{ in } (\nu + 9)$
- $((\textit{reset } x \textit{ in } \nu) + 9)(x) = 9$

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- For a set $\mathcal{C}$ of clocks, $\nu \in V_{\mathcal{C}}$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in \mathcal{C}$.
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$$(\textit{reset } x \textit{ in } \nu)(y) = \begin{cases} \nu(y) & \textit{if } y \neq x \\ 0 & \textit{else} \end{cases}$$

What does it mean?

- $\nu + 9$
- *reset* x *in* $(\nu + 9)$
- (*reset* x *in* $\nu) + 9$
- *reset* x *in* (*reset* y *in* ν)

$\nu(x)$		$\nu(x) = 0$
0	9	$\nu \not\models x \geq 2$
0	0	$\nu + 2 \models x \geq 2$
0	9	$\nu + 3 \models x \geq 2$
0	0	reset x in $\nu \not\models x \geq 2$

Timed automata

A timed automaton is a special hybrid automaton:

- All variables are clocks.
- Edges are defined by
  - source and target locations,
  - a label,
  - a guard: clock constraint specifying enabling,
  - a set of clocks to be reset.
- Invariants are clock constraints.

Timed automaton

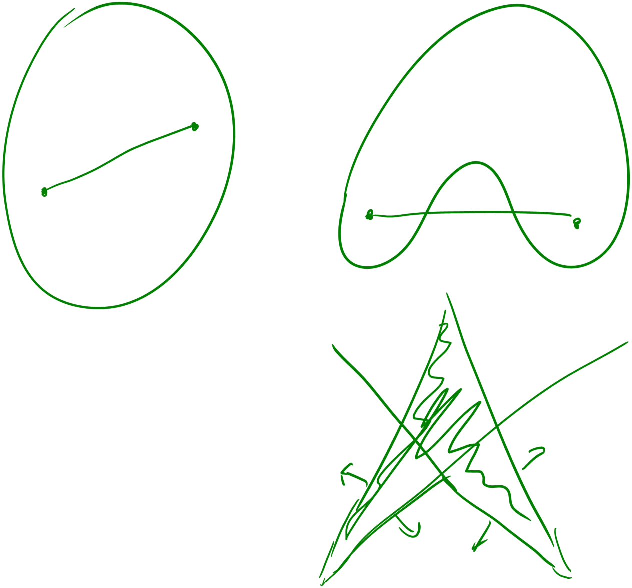
Definition (Syntax of timed automata)

A timed automaton $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$ is a tuple with

- Loc is a finite set of locations,
- $\mathcal{C}$ is a finite set of clocks,
- Lab is a finite set of synchronization labels,
- $Edge \subseteq Loc \times Lab \times (CC(\mathcal{C}) \times 2^{\mathcal{C}}) \times Loc$ is a finite set of edges,
- $Inv : Loc \to CC(\mathcal{C})$ is a function assigning an invariant to each location, and
- $Init \subseteq \Sigma$ with $\nu(x) = 0$ for all $x \in \mathcal{C}$ and all $(l, \nu) \in Init$.

We call the variables in $\mathcal{C}$ clocks. We also use the notation $l \overset{a:g,C}{\hookrightarrow} l'$ to state that there exists an edge $(l, a, (g, C), l') \in Edge$.

Note: (1) no explicit activities given (2) restricted logic for constraints



Timed automaton

Analogously to Kripke structures, we can additionally define

- a set of atomic propositions AP and
- a labeling function $L : Loc \to 2^{AP}$

to model further system properties.

Operational semantics

$$\frac{(l, a, (g, \mathcal{R}), l') \in Edge \qquad \nu \models g \quad \nu' = \textit{reset } \mathcal{R} \textit{ in } \nu \quad \nu' \models Inv(l')}{(l, \nu) \xrightarrow{a} (l', \nu')} \quad \texttt{Rule}_{\texttt{Discrete}}$$

$$\frac{t > 0 \quad \nu' = \nu + t \quad \nu' \models Inv(l)}{(l, \nu) \xrightarrow{t} (l, \nu')} \quad \texttt{Rule}_{\texttt{Time}}$$

Operational semantics

$$\frac{(l, a, (g, \mathcal{R}), l') \in Edge \qquad \nu \models g \quad \nu' = \textit{reset } \mathcal{R} \textit{ in } \nu \quad \nu' \models Inv(l')}{(l, \nu) \xrightarrow{a} (l', \nu')} \quad \texttt{Rule}_{\texttt{ Discrete}}$$

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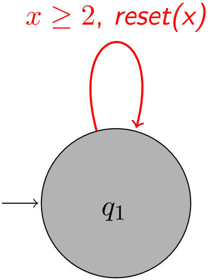
- Execution step: $\rightarrow \; = \; \xrightarrow{a} \cup \xrightarrow{t}$
- Path: $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \nu_0)$ and $\nu_0 \in Inv(l_0)$
- Initial path: path $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots$ with $\sigma_0 = (l_0, \nu_0)$, $l_0 \in Init$ and $\nu_0(x) = 0$ f.a. $x \in \mathcal{C}$
- Reachability of a state: exists an initial path leading to the state

Example: Timed Automaton



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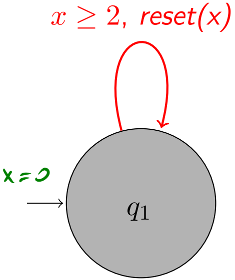

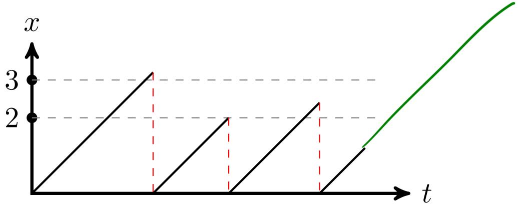

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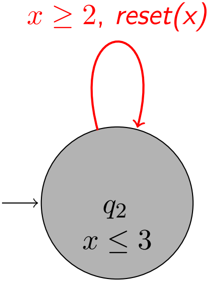
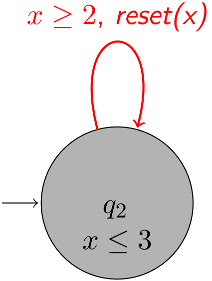
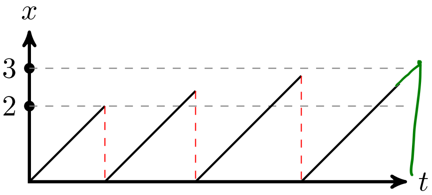
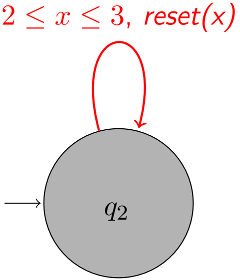
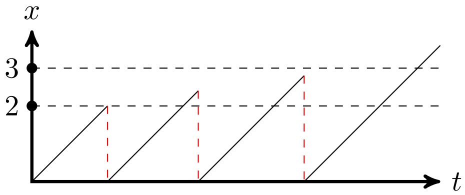



$x \geq 2$, reset(x)
q_2
$x \leq 3$

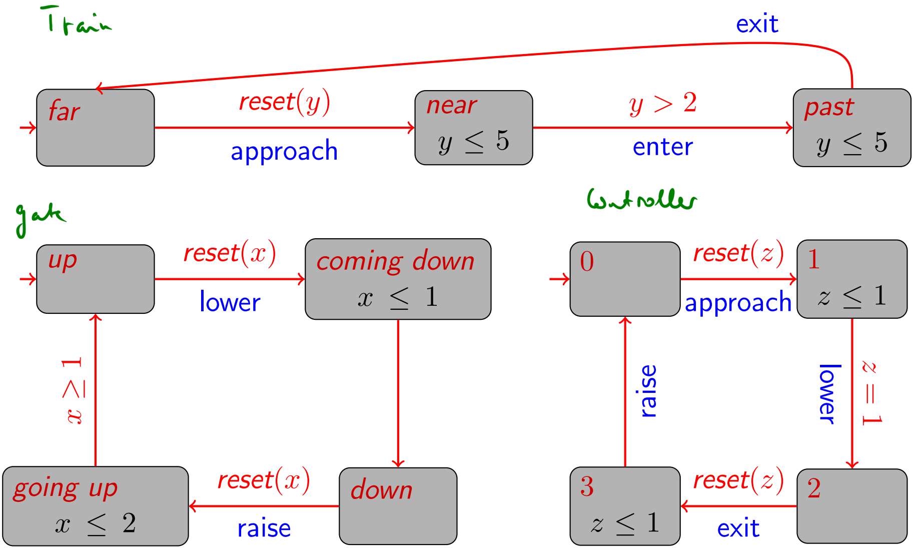

x
3
2
t

Example: Timed Automaton

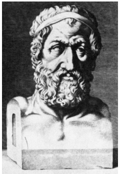




Example: Railroad Crossing

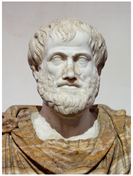


Time divergence, timelock, and zenoness



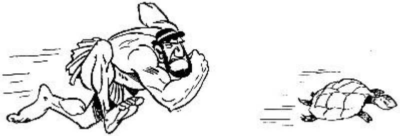
Zeno of Elea

(ca.490 BC-ca.430 BC)



Aristotle

(384 BC-322 BC)

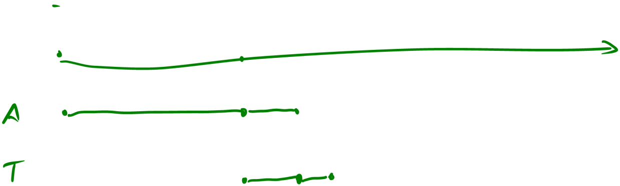


Paradox:
Achilles and the tortoise

(Achilles was the great Greek hero of Homer's The Iliad.)

"In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point where the pursued started, so that the slower must always hold a lead."
–Aristotle, Physics VI:9, 239b15

- Not all paths of a timed automata represent realistic behaviour.
- Three essential phenomena: time convergence, timelock, zenoness.


A
T

Time convergence

Definition

For a timed automaton $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$. we define $ExecTime : (Lab \cup \mathbb{R}^{\geq 0}) \to \mathbb{R}^{\geq 0}$ with

- $ExecTime(a) = 0$ for $a \in Lab$ and
- $ExecTime(d) = d$ for $d \in \mathbb{R}^{\geq 0}$.

Furthermore, for $\rho = s_0 \stackrel{\alpha_0}{\to} s_1 \stackrel{\alpha_1}{\to} s_2 \stackrel{\alpha_2}{\to} \ldots$ we define

$$ExecTime(\rho) = \sum_{i=0}^{\infty} ExecTime(\alpha_i).$$

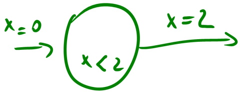
A path is time-divergent iff $ExecTime(\rho) = \infty$, and time-convergent otherwise.

- Time-convergent paths are not realistic, and are not considered in the semantics.
- Note: their existence cannot be avoided (in general).

$\pi : \bullet \xrightarrow{1} \bullet \xrightarrow{1/2} \xrightarrow{1/4} \xrightarrow{1/8} \ldots$

$$\text{ExecTime}(\pi) = \sum_{i=1}^{\infty} \frac{1}{i} = 2$$

Timelock



Definition

For a state $\sigma \in \Sigma$ let $Paths_{div}(\sigma)$ be the set of time-divergent paths starting in σ.
A state $\sigma \in \Sigma$ contains a timelock iff $Paths_{div}(\sigma) = \emptyset$.
A timed automaton is timelock-free iff none of its reachable states contains a timelock.

Timelocks are modeling flows and should be avoided.

Zenoness

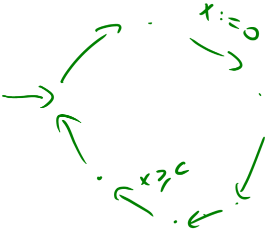
Definition

An infinite path fragment π is zeno iff it is time-convergent and infinitely many discrete actions are executed within π.
A timed automaton is non-zeno iff no zeno path starts in an initial state.

- Zeno paths represent nonrealizable behaviour, since their execution would require infinitely fast processors.
- Thus zeno paths are modeling flows and should be avoided.
- To check whether a timed automaton is non-zeno is algorithmically difficult.
- Instead, sufficient conditions are considered that are simple to check, e.g., by static analysis.

$$x = 0 \xrightarrow{a} x = 0 \xrightarrow{a} x = 0 \xrightarrow{a}$$

$$x = 0 \xrightarrow{1} x = 1 \xrightarrow{a} x = 0 \xrightarrow{1/2} x = \tfrac{1}{2} \xrightarrow{a} x = 0 \xrightarrow{1/4} x = \tfrac{1}{4} \xrightarrow{a} \ldots$$



Checking non-zenoness

Theorem (Sufficient condition for non-zenoness)

Let $\mathcal{T}$ be a timed automaton with clocks $\mathcal{C}$ such that for every control cycle

$$l_0 \stackrel{\alpha_1:g_1,C_1}{\longrightarrow} l_1 \stackrel{\alpha_2:g_2,C_2}{\longrightarrow} l_2 \ldots \stackrel{\alpha_n:g_n,C_n}{\longrightarrow} l_n = l_0$$

in $\mathcal{T}$ there exists a clock $x \in \mathcal{C}$ such that

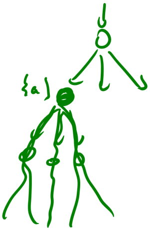
- $x \in C_i$ *for some* $0 < i \leq n$*, and*
- *for all evaluations* $\nu \in V$ *there exist some* $0 < j \leq n$ *and* $d \in \mathbb{N}^{>0}$ *with*

$$\nu(x) < d \quad \textit{implies} \quad (\nu \not\models Inv(l_j) \text{ or } \nu \not\models g_j).$$

Then $\mathcal{T}$ is non-zeno.

Contents

TCTL



- How to describe the behaviour of timed automata?
- Logic: TCTL, a real-time variant of CTL
- Syntax:

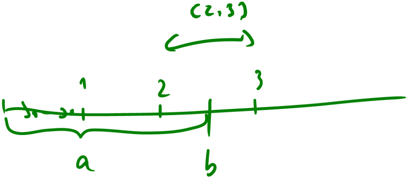
State formulae

$$\psi \quad ::= \quad \textit{true} \quad | \quad a \quad | \quad g \quad | \quad \psi \wedge \psi \quad | \quad \neg\psi \quad | \quad \mathbf{E}\varphi \quad | \quad \mathbf{A}\varphi$$

Path formulae:

$$\varphi \quad ::= \quad \psi\, \mathcal{U}^J\, \psi$$

with $J \subseteq \mathbb{R}^{\geq 0}$ is an interval with integer bounds (open right bound may be ∞).



$a \; \mathcal{U}^{[2,3]} \; b$

request $\mathcal{U}$ response

- Syntactic sugar:

$$\mathcal{F}^J \psi \quad := \quad \textit{true}\, \mathcal{U}^J\ \psi$$

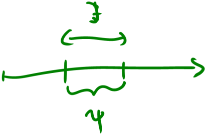
$$\mathbf{E}\mathcal{G}^J \psi \quad := \quad \neg \mathbf{A}\mathcal{F}^J \neg \psi$$

$$\mathbf{A}\mathcal{G}^J \psi \quad := \quad \neg \mathbf{E}\mathcal{F}^J \neg \psi$$

$$\psi_1\ \mathcal{U}\ \psi_1 \quad := \quad \psi_1\ \mathcal{U}^{[0,\infty)}\ \psi_2$$

$$\mathcal{F}\psi \quad := \quad \mathcal{F}^{[0,\infty)} \psi$$

$$\mathcal{G}\psi \quad := \quad \mathcal{G}^{[0,\infty)} \psi$$



- Note: no next-time operator

TCTL semantics

Definition (TCTL continuous semantics)

Let $\mathcal{T} = (Loc, \mathcal{C}, Lab, Edge, Inv, Init)$ be a timed automaton, AP a set of atomic propositions, and $L : Loc \to 2^{AP}$ a state labeling function. The function $\models$ assigns a truth value to each TCTL state and path formulae as follows:

$$
\begin{array}{lll}
\sigma & \models \mathit{true} & \\
\sigma & \models a & \textit{iff} \quad a \in L(\sigma) \\
\sigma & \models g & \textit{iff} \quad \sigma \models g \\
\sigma & \models \neg\psi & \textit{iff} \quad \sigma \not\models \psi \\
\sigma & \models \psi_1 \wedge \psi_2 & \textit{iff} \quad \sigma \models \psi_1 \textit{ and } \sigma \models \psi_2 \\
 & & \\
\sigma & \models \mathbf{E}\varphi & \textit{iff} \quad \pi \models \varphi \textit{ for some } \pi \in Paths_{div}(\sigma) \\
\sigma & \models \mathbf{A}\varphi & \textit{iff} \quad \pi \models \varphi \textit{ for all } \pi \in Paths_{div}(\sigma).
\end{array}
$$

where $\sigma \in \Sigma$, $a \in AP$, $g \in ACC(\mathcal{C})$, ψ, ψ_1 and ψ_2 are TCTL state formulae, and φ is a TCTL path formula.

TCTL semantics

Meaning of $\mathcal{U}$: a time-divergent path satisfies $\psi_1 \; \mathcal{U}^J \; \psi_2$ whenever at some time point in J property ψ_2 holds and at all previous time instants $\psi_1 \vee \psi_2$ is satisfied.

TCTL semantics

Definition (TCTL continuous semantics)

For a time-divergent path $\pi = \sigma_0 \stackrel{\alpha_1}{\rightarrow} \sigma_1 \stackrel{\alpha_2}{\rightarrow} \ldots$ we define $\pi \models \psi_1 \, \mathcal{U}^J \, \psi_2$ iff

- $\exists i \geq 0.\ \sigma_i + d \models \psi_1$ for some $d \in [0, d_i]$ with
$$(\sum_{k=0}^{i-1} d_k) + d \in J, \ \textit{and}$$
- $\forall j \leq i.\ \sigma_j + d' \models \psi_1 \vee \psi_2$ for any $d' \in [0, d_j]$ with
$$(\sum_{k=0}^{j-1} d_k) + d' \leq (\sum_{k=0}^{i-1} d_k) + d$$

where $d_i = ExecTime(\alpha_i)$.

Satisfaction set

Definition

For a timed automaton $\mathcal{T}$ with clocks $\mathcal{C}$ and locations Loc, and a TCTL state formula ψ the satisfaction set $Sat(\psi)$ is defined by

$$Sat(\psi) = \{s \in \Sigma | s \models \psi\}.$$

$\mathcal{T}$ satisfies ψ iff ψ holds in all initial states:

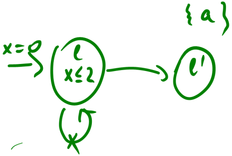
$$\mathcal{T} \models \psi \textit{ iff } \forall l_0 \in Init.\ (l_0, \nu_0) \models \psi$$

$\forall x \in C .\ \nu_0(x) = 0$

where $\nu_0(x) = 0$ for all $x \in \mathcal{C}$.

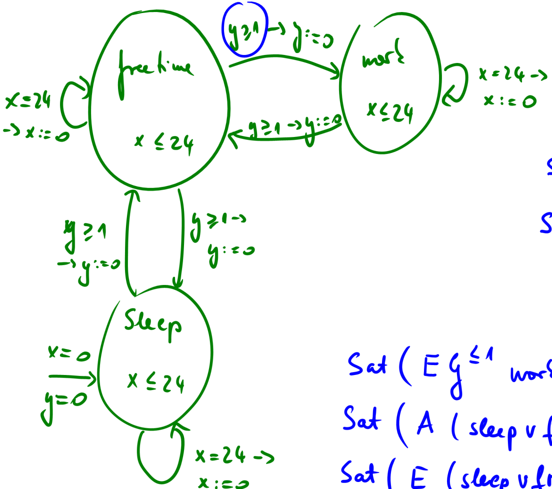
TCTL vs. CTL

- TCTL formulae with intervals $[0, \infty)$ may be considered as CTL formulae
- However, there is a difference due to time convergent paths
- TCTL ranges over time-divergent paths, whereas CTL over all paths!



$AF\,a$

$$\rightarrow (\ell, x=0) \xrightarrow{1} (\ell, x=1) \xrightarrow{1/2} (\ell, x=1{,}5) \xrightarrow{1/4} (\ell, x=1{,}75) \rightarrow \ldots$$



$Sat(AF\, work) = \{work\} \times V$

$Sat(EF\, work) = Loc \times V = \Sigma$

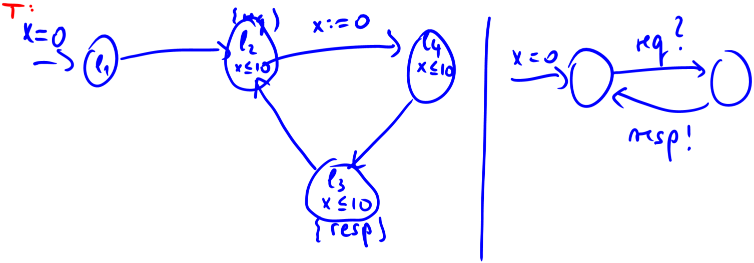
$Sat(EF^{\leq 2}\, work) = \Sigma$

$Sat(EF^{\leq 1}\, work) = \{(\ell, \nu) \in \Sigma \mid \ell = work \vee \ell = freetime \vee (\ell = sleep \wedge y \geq 1)\}$

$Sat(EG^{\leq 1}\, work) = \{(\ell, \nu) \in \Sigma \mid \ell = work \wedge y = 0\}$

$Sat(A\, (sleep \vee freetime)\, \mathcal{U}\, work) = \{work\} \times V$

$Sat(E\, (sleep \vee freetime)\, \mathcal{U}^{[1,2]}\, work) = \Sigma$



$Sat(AG\,AF\,req) = \{\ell_2, \ell_3, \ell_4\} \times V$

$(\ell_1, x=0) \xrightarrow{0} (\ell_2, x=0)$

$\downarrow 10$

$(\ell_4, x=0) \xleftarrow{0} (\ell_2, x=10)$

$\downarrow 10$

$(\ell_4, x=10) \xrightarrow{0} (\ell_3, x=10)$

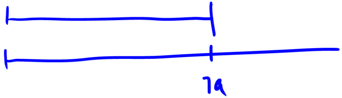
$Sat(AF\,req) =$

$\{\ell_2, \ell_3, \ell_4\} \times V$

$Sat(AG(req \rightarrow AF^{\leq 20} resp))$

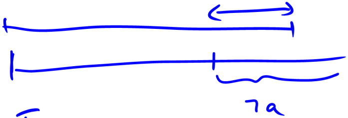
$AG((req \wedge x \geq 10) \rightarrow AF^{\leq 10} resp)$

TF – 11 –



$G\,a$ — safety

violation: finite path



$G F a$ — liveness

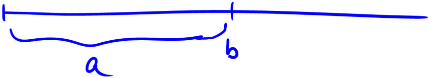
violation: infinite path

$F a$ — violation:

$F G a$ — liveness

violation:

$a\, U\, b$



violation: i) $G \neg b$ — safety

ii)

	safety	liveness
witness	infinite	finite
⇒ counter-example	finite	infinite

$$a \, \mathcal{U} \, b = a \, \mathcal{U}_{weak} \, b \wedge F b$$

$a \, \mathcal{U}_{weak} \, b$ ↑ safety

$F b$ ↑ liveness